J. Partial Differential Equations
Vol. 3, No. 1(1990), 47—68

# RADIAL SOLUTIONS OF FREE BOUNDARY PROBLEMS FOR DEGENERATE PARABOLIC EQUATIONS

Li Huilai
(Institute of Math., Jilin University)
(Received May 5, 1988; revised April 10, 1989)

**Abstract** In this paper we are devoted to the free boundary problem

$$\begin{cases} u_t = \Delta A(u) & (x,t) \in G_{\Gamma,T} \\ u(x,0) = \varphi(x) & x \in G_0 \\ u|_\Gamma = 0 & \\ (\frac{\partial A(u)}{\partial x_i}\nu_i + \psi(x)\nu_t)|_\Gamma = 0 & \end{cases}$$

where $A'(u) \geqslant 0$. Under suitable assumptions we obtain the existence and uniqueness of global radial solutions for $n=2$ and local radial solutions for $n \geqslant 3$.

**Key Words** High dimensions; degenerate parabolic; free boundary.
**Classification** 35K65.

## 1. Introduction

This paper is devoted to the following free boundary problem

$$\begin{cases} u_t = \Delta A(u) & (x,t) \in G_{\Gamma,T} \\ u(x,0) = \varphi(x) & x \in G_0 \\ u|_\Gamma = 0 & \\ (\frac{\partial A(u)}{\partial x_i}\nu_i + \psi(x)\nu_t)|_\Gamma = 0 & \end{cases} \tag{1.1}$$

where $G_0$ is a domain in $R^n$, $A'(u) \geqslant 0$, $\Gamma$ is a surface in $R^n \times (0,T)$, $G_{\Gamma,T}$ is the domain bounded by $G_0$, $\Gamma$, and $\{t=T\}$, and $(\nu_x; \nu_t)$ is the normal to $\Gamma$.

The problem (1.1) comes from the analysis of the structure of discontinuous solutions for the equation $u_t = \Delta A(u)$ (see [7]). We also remark that the free boundary problem (1.1) is, in its form, the so-called Stefan problem studied by many authors. The difference between the problem (1.1) and Stefan problem is the degenerality in (1.1).

In the case $n=1$, we have dealt thoroughly with the problem (1.1). Under very general assumptions on $A(u)$, $\varphi$ and $\psi$, we proved the existence and uniqueness of (1.1) and discussed the smoothness of free boundary. We obtained the necessary and sufficient condition for the free boundary in $C^1$ (see [4]).

We will restrict our attention to the problem (1.1) for $n \geqslant 2$ in this paper and only discuss a special case that can be reduced to a one-dimensional problem. The fundamental assumptions are

(H) $A(u)=u^m$ ($m>1$ is a constant), $G_0=B_1(O)$ is the unit ball in $R^n$, $\varphi(x)=\varphi(r)$, $\psi(x)=\psi(r)$, $r=(x_1^2+\cdots+x_n^2)^{1/2}$.

If the solution $u$ has the form $u(x,t)=u(r,t)$ and $\Gamma$ is determined by the function $r=\lambda(t)$ ($\lambda(0)=1$), then as a function of $(r,t)$, $(u,\lambda)$ satisfies

$$\begin{cases} u_t = u_{rr}^m + \dfrac{n-1}{r}u_r^m & 0<r<\lambda(t), 0<t<T \\ u(r,0) = \varphi(r), & 0<r<1 \\ u(\lambda(t),t) = 0, & 0<t<T \\ u_r^m(\lambda(t),t) = \psi(\lambda(t))\lambda'(t), & 0<t<T \end{cases} \tag{1.2}$$

It is worth remarking that when $n=1$, the problem (1.2) has only one kind of degenerality, but for $n\geqslant 2$, besides the degenerality of $u^m$, there is an irregular factor $1/r$, which results in the important difference between $n=1$ and $n\geqslant 2$.

To solve (1.2), we introduce the transform:

$$y=\frac{r^{2-n}}{n-2} \quad (n>2), \qquad y=-\ln r \quad (n=2)$$

Set $v(y,t)=u(r,t)$, then $v(y,t)$ satisfies

$$\begin{cases} v_t = g_n(y)v_{yy}^m, & \lambda_n(t)<y<\infty, 0<t<T \\ v(y,0) = \varphi_n(y), & a_n<y<\infty \\ v(\lambda_n(t),t) = 0, & 0<t<T \\ v_y^m(\lambda_n(t),t) = \psi_n(\lambda_n(t))\lambda_n'(t), & 0<t<T \end{cases} \tag{1.3}$$

where for $n>2$,

$$g_n(y) = (n-2)^{2(n-1)/(n-2)}y^{2(n-1)/(n-2)}, \varphi_n(y) = \varphi(((n-2)y)^{1/(2-n)})$$

$$a_n = \frac{1}{n-2}, \lambda_n(t) = \frac{\lambda(t)^{n-2}}{n-2}$$

$$\psi_n(y) = (n-2)^{-2(n-1)/(n-2)}\psi(((n-2)y)^{1/(2-n)})y^{-2(n-1)/(n-2)}, 0<y<a_n$$

and for $n=2$,

$$g_n(y) = e^{2y}, \quad \varphi_n(y) = \varphi(e^{-y}), a_n = 0$$

$$\lambda_n(t) = -\ln\lambda(t), \quad \psi_n(y) = e^{-2y}\psi(e^{-y})$$

The paper is arranged as follows: In Section 2, we study the existence, uniqueness and regularity of the solutions of the problem (1.2). We prove that if $\varphi$ and $\psi$ satisfy suitable conditions, then for $n=2$, the problem (1.2) has a unique soluiton $u$ for any $T>0$, and for $n\geqslant 3$ there exists a constant $t_n>0$ such that the problem (1.2) has a (unique) solution in $(0,t_n)$. In Section 3, we turn the results for the problem (1.2) to (1.1). The key to this procedure is to prove the following conclusion

$$\lim_{y\to\infty}(g_n(y))^\alpha\int_0^1|u_y^m(y,s)|ds = 0$$

where $\alpha<n/(2(n-1))$ ($n\geqslant 2$), and $u$ is the solution of the problem (1.2). The uniqueness of the problem (1.1) can be obtained as a consequence of a result due to Brézis and Crandall [1].

Nevertheless we do not obtain the condition for the free boundary in $C^1$ for $n\geqslant 2$.