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SINGULARITIES OF SOLUTIONS TO CAUCHY PROBLEMS FOR SEMILINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS

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Abstract This paper concerns the Cauchy problem for semilinear wave equations with two space variables, of which the initial data have conormal singularities on finite curves intersecting at one point on the initial plane. It is proved that the solution is of conormal distribution type, and its singularities are contained in the union of the characteristic surfaces through these curves and the characteristic cone issuing from the intersection point.

Key Words Semilinear wave equations; propagation of singularities; conormal distributions.

Classifications 35L70; 35B65.

1. Introduction

Consider the semilinear Cauchy problem

$$Pu = f(t,x,u), \qquad (t,x) \in \Omega \subset R^{1+2} \tag{1.1}$$

$$u(0,x) = g(x), \qquad \partial_t u(0,x) = h(x), \quad x \in \Omega \cap \{t=0\} \subset R^2 \tag{1.2}$$

for a second order strictly hyperbolic partial differential operator $P=P(t,x,\partial_t,\partial_x)$. We are concerned with the singularities of the solution u knowing the singularities of the Cauchy data g and h.

There has been a considerable amount of work in this direction, e.g. Bony [1], Ritter [10], Rauch and Reed [9], Metivier [6,7] and other papers cited there. The results applied to (1.1)—(1.2) show that the singularities of u will lie in the characteristic surfaces through the curve to which conormal singularities of g or h are confined. In case there are more than one such curves present in $\Omega \cap \{t=0\}$, the singularities of u may spread to the characteristic cones issuing from the intersection points of these curves, as was illustrated in an example given by Rauch and Reed [8]. The relevant analyses to treat this type of phenomenon were later carried out independently and by different methods in Bony [2] and in Melrose and Ritter [4], where the singularities of the solution u to (1.1) for $t>0$ were studied knowing its singularities for $t<0$.

We study the case where the Cauchy data g and h have conormal singularities along finite C^∞ curves which can intersect each other transversally. The result shows that, locally in t, the singularities of the solution u to (1.1)—(1.2) are localized in the characteristic surfaces through these curves and in the characteristic cones from the intersection points of these curves. Bony [3] has announced a similar result which assumes $u \in H^s, s > 3/2$, and deals with weak singularities. In our result, besides the difference of the methods, it is assumed only $u \in L^\infty$ and the Cauchy datum h may have jump discontinuities (i. e. strong singularities) over curves.

The spaces of conormal distributions used to describe the singularities of solutions are equivalent to the spaces in [2] and [4]. We prove the main theorem by an improvement of the approach in [4], where there is loss of derivatives for the result. This defect is overcome in our treatment.

2. Notations and Statement of the Result

For the sake of simplicity, we will state and prove our result only for the special wave operator $\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2$. The proof in this paper is valid for general second order strictly hyperbolic operators with C^∞ coefficients.

Let Ω be a bounded region of R^3 containing $O, \omega = \Omega \cap \{t=0\}$, and

$$P = \partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 \tag{2.1}$$

Suppose Ω is a domain of determinacy of ω with respect to P. Let $C_i, i=1, \cdots, N, N > 1$, be C^∞ curves in ω intersecting transversally at one point, say $O=(0,0,0)$. We assume that there is no other intersection of any two of these curves. The two characteristic surfaces through C_i are denoted by $S_i, S_{N+i}, i=1, \cdots, N$, the characteristic cone from O is

$$S_0 = \{(t, x_1, x_2) : t^2 - x_1^2 - x_2^2 = 0\}$$

It is assumed that $S_j, j=0, \cdots, 2N$, are all regular C^∞ surfaces in $\Omega \backslash O$ and have no triple intersection there, otherwise we can shrink Ω.

Following the notations in [4], [5], for any Lie algebra $\mathscr{V}_*$ of vector fields (i. e. homogeneous first order differential operators) on a C^∞ manifold $M \subset R^n$, we define the associated space of conormal distributions

$$I_k L^p(M, \mathscr{V}_*) = \{v \in L^p(M) : V_1 \cdots V_i v \in L^p(M)$$
$$\text{for any } \; V_j \in \mathscr{V}_*, 1 \leqslant j \leqslant i \leqslant k\}, \quad (1 < p < \infty)$$

For any finite collection $\mathscr{G}$ of C^∞ submanifolds of M, define the Lie algebra of C^∞ vector fields

$$\mathscr{V}(\mathscr{G}) = \{V \in C^\infty(M, TM) : V \text{ is tangent to each submanifold in } \mathscr{G}\}$$

Now let

$$\mathscr{C}_i = \{C_i \backslash O, O\}, \quad i = 1, \cdots, N$$
$$\mathscr{S}_{ij} = \{S_i \backslash O, S_j \backslash O, O\}, \quad 0 \leqslant i < j \leqslant 2N$$

The main theorem which will be proved in Section 6 is

Theorem 2.1 *Suppose $u \in L^\infty$ is a solution to the Cauchy problem*

$$Pu = f(t, x, u), \qquad (t,x) \in \Omega \tag{2.2}$$
$$u(0,x) = g(x), \qquad \partial_t u(0,x) = h(x), \qquad x \in \omega \tag{2.3}$$

where $f \in C^\infty(\bar{\Omega} \times C^1)$, P is given by (2.1). If for integer $0 \leqslant k \leqslant \infty$, there exists some $p > 2$ such that