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BOUNDARY-VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE①

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Abstract In this paper we study the existence of solutions to the Dirichlet problem for a class of integro-differential equations of elliptic type by using the weakly continuous method.

Key Words Integro-differential equations; weakly continuous operator; Choquard equation; weak solutions.

Classifications 45K05; 35J60.

0. Introduction

The integro-diffferential equations of elliptic type occur in many practical models in nuclear physics, theory of quantum field and mechanics.

Ugowski [1] and Tsai Longyi [2] considered the following problem

$$a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x,u,K(u)), \quad x \in \Omega \tag{0.1}$$

$$u|_{\partial\Omega} = \varphi(x) \tag{0.2}$$

where $K(u)$ denotes an integral operator, and $\Omega \subset R^m$ is a bounded region.

Ugowski discussed the existence of (0.1), (0.2) by using a successive approximation. Tsai Longyi discussed the existence of (0.1), (0.2) by combining methods of supersolution-subsolution and topological degree. Politjukov [3] defined a concept concerning ε-supersolution and ε-subsolution, and discussed parabolic equations by using this method.

What we shall discuss is the following problem

$$\sum_{|\alpha|,|\beta|=n} (-1)^{\alpha} D_\alpha(a_{\alpha,\beta}(x, \Lambda u, R(u))D_\beta u) + \sum_{|\gamma|\leqslant n} (-1)^{|\gamma|} D_\gamma b_\gamma(x, \Lambda u, R(u)) = 0, \quad x \in \Omega \tag{0.3}$$

$$D_\gamma u|_{\partial\Omega} = 0, \quad \forall\ |\gamma| \leqslant n-1 \tag{0.4}$$

where $\Lambda u = (D_\gamma u, |\gamma| \leqslant n-1)$, $R(u)$ is an integral operator acting on Λu, and $\Omega \subset R^m$ is an arbitrary region.

1. The Existence Theorem of the Weakly Continuous Operator Equations

Let X be a linear space, X_1, X_2 be the completions of X with respect to the norm

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$\|\cdot\|_1$ and $\|\cdot\|_2$ respectively, X with respect to $\|\cdot\|_2$ be a separable linear normed space. X_1 be a reflexive Banach space. $x_n \rightharpoonup x_0$ denotes weak convergence and $x_n \to x_0$ denotes strong convergence.

Definition 1.1 *A mapping $G: X_1 \to X_2^*$ is called weakly continuous if for any $x_n, x_0 \in X_1, x_n \rightharpoonup x_0$, there is*

$$\lim_{n\to\infty} \langle Gx_n, y\rangle = \langle Gx_0, y\rangle, \quad \forall\, y \in X_2$$

Theorem 1.2 *Let $G: X_1 \to X_2^*$ be a weakly continuous mapping. If there exists a bounded open set Ω of $X_1, O \in \Omega$, such that*

$$\langle Gu, u\rangle \geqslant 0, \quad \forall\, u \in \partial\Omega \cap X \tag{1.1}$$

then $Gu=0$ has a solution u_0 in X_1, and $u_0 \in \overline{\mathrm{co}\Omega}$.

Proof Take $\{e_i\} \subset X$, such that it is dense in X_2, and denote $\widetilde{X}_n = \mathrm{span}\{e_1, \cdots, e_n\}$, $\widetilde{X}_n$ has the same norm as that of X_1. Define the mapping $A_n: \widetilde{X}_n \to \widetilde{X}_n^*$ as

$$\langle A_n u, v\rangle = \langle Gu, v\rangle, \quad \forall\, u, v \in \widetilde{X}_n$$

It is easy to derive the continuity of A_n from the weak continuity of G. By (1.1) we have

$$\langle A_n u, u\rangle = \langle Gu, u\rangle \geqslant 0, \quad \forall\, u \in \partial\Omega \cap \widetilde{X}_n$$

Using the acute angle principle [4] of the topological degree, there exists $u_n \in \overline{\Omega} \cap \widetilde{X}_n$ such that $\langle A_n u_n, v\rangle = \langle Gu_n, v\rangle = 0, \quad \forall\, v \in \widetilde{X}_n$.

Since $\{u_n\}$ is bounded in X_1 and X_1 is reflexive, let, say, $u_n \rightharpoonup u_0 \in X_1$, hence it follows that

$$\lim_{k\to\infty} \langle Gu_k, v\rangle = \langle Gu_0, v\rangle = 0, \quad \forall\, v \in \widetilde{X}_n$$

Because $\bigcup_n \widetilde{X}_n$ is dense in X_2, we have

$$\langle Gu_0, v\rangle = 0, \quad \forall\, v \in X_2$$

i. e., $Gu_0 = 0$. Therefore the theorem is proved.

2. The Elliptic Dirichlet Problem

We consider the following problem

$$\sum_{|\alpha|, |\beta| = n} (-1)^n D_\alpha (a_{\alpha,\beta}(x, \Lambda u, R(u)) D_\beta u) + \sum_{|\gamma| \leqslant n} (-1)^{|\gamma|} D_\gamma b_\gamma (x, \Lambda u, R(u)) = f(x), \quad x \in \Omega \tag{2.1}$$

$$D_\gamma u|_{\partial\Omega} = 0, \quad |\gamma| \leqslant n-1 \tag{2.2}$$

where $\Lambda u = \{D_\alpha u \,|\, |\alpha| \leqslant n-1\}$, $R(u)$ is an integral operator acting on Λu and $\Omega \subset R^m$ is any region.

First of all, some comments must be made for the related notations of the anisotropic Sobolev space. We denote

$$W^{p_\alpha}_{|\alpha|\leqslant k}(\Omega) = \{u \in L^{p_0}(\Omega), p_0 \geqslant 1 \,|\, D_\alpha u \in L^{p_\alpha}(\Omega), |\alpha| \leqslant k, p_\alpha \geqslant 1 \text{ or } p_\alpha = 0\}$$

with the norm

$$\|u\| = \sum_{|\alpha|\leqslant k} \mathrm{sign}\; p_\alpha \|D_\alpha u\|_{L^{p_\alpha}}$$