J. Partial Differential Equations
Vol. 3, *No.* 2(1990), 63—76

GEVREY-HYPOELLIPTICITY FOR A CLASS OF PARABOLIC TYPE OPERATORS[①]

Chen Hua
(Dept. of Math., Wuhan Univ.)

(Received May 9, 1988; revised January 24, 1989)

Abstract This paper studies the micro-local version of the Gevrey-hypoellipticity for a class of parabolic type operator, $\partial_t - a(t,x;D_x)$ $(a(t,x;\xi)\in C^\infty([0,T], S^m_{\sigma^s}(R^n))$ $(m\geqslant 1/s)$ is a Gevrey-pseudoanalytic symbol of class $s(s>1)$), and we obtain the following main result: Under the condition (I), the operator stated above is micro-local Gevrey-hypoelliptic. In order to prove our main result, the auther have used (α,β) method in this paper.

Key Words Gevrey-hypoellipticity; wave front set in Gevrey-class; (α,β) method.

Classification 35H05.

1. Statement of Main Result

Let us consider the micro-local version of the Gevrey-hypoellipticity (more precisely Gevrey-hypoellipticity in x) for a class of parabolic type operators

$$\partial_t - a(t,x;D_x) \tag{1.1}$$

where $a(t,x;\xi)\in C^\infty([0,T], S^m_{\sigma^s}(R^n))$ $(m\geqslant 1/s)$ is a Gevrey-pseudoanalytic symbol of class $s(s>1)$, t is considered as a parameter. If $a^{(\alpha)}_{(\beta)}(t,x;\xi)=\partial^\alpha_\xi D^\beta_x a(t,x;\xi)$ $(\alpha,\beta\in Z^n_+)$, then we have the following estimate:

$$|a^{(\alpha)}_{(\beta)}(t,x;\xi)|\leqslant C_0\alpha!(\beta!)^s C^{|\alpha+\beta|}|\xi|^{m-|\alpha|},\quad \text{for } |\alpha|\leqslant R^{-1}|\xi|^{1/s},\quad t\in[0,T] \tag{1.2}$$

where R is a suitable large number, C_0 and C are constants.

Let $a_m(t,x;\xi)$ be the principal symbol of $a(t,x;D_x)$, then condition

$$\operatorname{Re} a_m(0,x;\xi)\leqslant 0,\quad (x,\xi)\in R^n\times R^n\backslash\{0\} \tag{1.3}$$

is necessary for well-posedness of the Cauchy problem for the operator (1.1) (see Mizohata [2]). Now we assume

(I) At $(x_0,\xi^0)\in R^n\times R^n$, $(|\xi^0|=1)$, $\exists\ \delta>0$, such that

---

① Supported by the Natural Science and Young Science Foundation of Academia Sinica.

$$\mathrm{Re}\, a_m(t,x_0;\xi^0) \leqslant -\delta , \; t \in [0,T]$$

Then from [2], we know the operator (1.1) is locally solvable. Suppose that $u(t,x) \in C^1([0,T],D^{(s)'})$ is a local solution of equation

$$\partial_t u(t,x) = a(t,x;D_x)u(t,x) + f(t,x) \tag{1.4}$$

where $D^{(s)'}$ is the ultradistribution space of class s, and $f(t,x) \in C([0,T],D^{(s)'})$. Then we have the following main result.

Theorem *Under condition* (I), *if* $(x_0,\xi^0) \overline{\in} WF_s(f(\cdot,t)), \forall\, t\in[0,T]$, *then*

$$(x_0,\xi^0) \overline{\in} WF_s(u(\cdot,t)) \quad \forall\, t \in (0,T],$$

Here, the definition of the wave front set in Gevrey class $WF_s(u)$ is given in the following way:

Definition 1.1 $(x_0,\xi^0) \overline{\in} WF_s(u)$, *if and only if there exists a cut-off function* $\psi(x) \in G^s \cap C_0^\infty$, *taking the value* 1 *in a neighborhood of* x_0 *such that the estimate*

$$|\widehat{\psi u}(\xi)| \leqslant \exp(-\varepsilon_0|\xi|^{1/s}), (\exists\, \varepsilon_0 > 0) \tag{1.5}$$

holds when ξ *tends to* ∞ *remaining in a suitable conic neighborhood* V_{ξ^0} *of* ξ^0.

By G^s-pseudolocal property (see [4]), $WF_s(f(\cdot,t)) \subset WF_s(u(\cdot,t))$. So if $\mathrm{Re}\, a_m(t,x;\xi)$ is strictly negative on $[0,T] \times R^n \times S^{n-1}$, then the above theorem gives

$$WF_s(f(t,\cdot)) = WF_s(u(t,\cdot)), \quad \forall\, t \in (0,T] \tag{1.6}$$

Let Π be the projection map $(x,\xi) \to x$, then we know $\Pi(WF_s(u)) = \mathrm{sing\,supp}_s\, u$ (i. e. Gevrey singular support of u), so (1.6)' implies

$$\mathrm{sing\,supp}_s\, (u(t,\cdot)) = \mathrm{sing\,supp}_s\, (f(t,\cdot)), \quad \forall\, t \in (0,T] \tag{1.7}$$

We have therefore obtained the following obvious corollary:

Corollary 1.1 *Under the above condition, the operator* (1.1) *is Gevrey-hypoelliptic.*

The main theorem also proves the microlocal Gevrey-hypoellipticity of elliptic operators.

Corollary 1.2 *Let* $a(x,D_x) \in S^m_{\sigma^s}$ *be an elliptic operator, then*

$$WF_s(au) = WF_s(u) \tag{1.8}$$

Proof By G^s-pseudolocal property, $WF_s(au) \subset WF_s(u)$. Let $au = f(x)$, then $u(x)$ satisfies the equation

$$\partial_t u = -(\bar{a}a)u + \bar{a}(x,D_x)f(x) \tag{1.9}$$