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THE HOPF BIFURCATION FOR AN ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATION AND ITS APPLICATION

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Abstract In this paper we obtained the Hopf bifurcation theorem for an abstract functional differential equation by the results of [1]. The asymptotic expression of bifurcation formulae and stability condition were given in detail. Applying the result, we considered the Hopf bifurcation problem for a reaction—diffusion equation with time delay.

Key words Hopf bifurcation; Functional differential equation
Classifications 34K30; 35B22; 35K57.

1. Introduction

There are a variety of mathematical models in electronics, ecology, biology and biochemistry etc., which are described by differential equations with time delay. Many problems of these models possess oscillatory phenomena or periodic oscillation. The research of this aspect was confined to ordinary differential equation with delay so far in a substantial literature, we refer the readers to [6], [7], [8], [9]. There are few results in partial differential equation with delay. In paper [2] and [3], the Hopf bifurcation of a semilinear diffusion equation with time delay arising in ecology was studied. In paper [1] we proved a Hopf bifurcation theorem for an abstract evolution equation by means of the center manifold reduction method. This paper is a continuous research of [1], we considered the Hopf bifurcation problem for an abstract functional differential equation here. Applying our abstract results, we considered a reaction—diffusion equation, more precisely, we considered the following problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + u(x,t) - \left(\dfrac{\pi}{2} + \mu\right)u(x,t-1)(1 + u(x,t)) \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

$$(x,t) \in (0,\pi) \times R^+$$

2. Preliminaries

Let X be a real Banach space. We shall still denote $X_C = X + iX$ by X if it is not confused. We consider the functional differential equation:

$$\frac{du}{dt} = A(\mu)u + \int_{-1}^{0} d\eta(\mu,\theta)u_t + f(\mu,u_t) \tag{1}$$

where $A(\mu)$ is a closed, densely defined linear operator on X and is the infinitesimal generator of a strongly continuous semigroup $T(t)$, and $T(t)$ is compact for each $t \geqslant 0$. $A(\mu) \in C^{L+2}$ in $\mu (L \geqslant 2)$.

Let $C = C([-r,0],X)$ be the Banach space of continuous X—valued function on $[-r,0]$, with supremum norm, where $r > 0$. If u is a continuous function from $[a-r,b]$ to X and $t \in [a,b]$, then u_t denotes the element of C given by $u_t(\theta) = u(t+\theta)$, $-r \leqslant \theta \leqslant 0$. μ is a real parameter. $\eta(\mu,\theta): R \times [-r,0] \to B(X,X)$ is of bounded variation, where $B(X,X)$ denotes the space of bounded linear everywhere defined operators from X to X, $\eta(\mu,\theta) \in C^{L+2} (L \geqslant 2)$ in μ. $f: R \times C \to X$ belongs to $C^{L+2} (L \geqslant 2)$ satisfying $f(\mu,0) = 0, D_u f(\mu,0) = 0$.

We refer the readers to [4] and [5] for the results of the existence, uniqueness, smoothness of the solution of (1) with initial value.

Define

$$(A_U(\mu)\phi)(\theta) = \begin{cases} \dfrac{d\phi}{d\theta} & -r \leqslant \theta < 0 \\ A(\mu)\phi(0) + \displaystyle\int_{-r}^{0} d\eta(\mu,\theta)\phi(\theta) & \theta = 0 \end{cases}$$

$$D(A_U(\mu)) = \{\phi \in C : \frac{d\phi}{d\theta} \in C, \phi(0) \in D(A(\mu))\}$$

For any $\phi \in C$, define $U(t)\phi = u_t(\phi)$, where $u_t(\phi)$ is the solution of (1) with initial value ϕ. From the result of [4] we know that $U(t)$ is a strongly continuous semigroup with infinitesimal generator $A_U(\mu)$, and $D(A_U(\mu))$ is densed in C.

Define

$$F(\mu,\phi)(\theta) = \begin{cases} 0 & -r \leqslant \theta < 0 \\ f(\mu,\phi(0)) & \theta = 0 \end{cases}$$

we can transform equation (1) into an abstract differential equation on C:

$$\frac{du_t}{dt} = A_U(\mu)u_t + F(\mu,u_t) \tag{2}$$

Define the linear operator $\Delta(\lambda): D(A_U) \to X$ by

$$\Delta(\lambda) = A(\mu) - \lambda I + \int_{-r}^{0} d\eta(\mu,\theta)e^{\lambda\theta}$$

We will say that λ satisfies the "characteristic equation" of (2) provided $\Delta(\lambda)\phi = 0$ for some $\phi \neq 0$.

Let $C^* = C([0,r],X^*)$, where X^* is the dual space of X. Also let $A(\mu)^*: X^* \to X^*$ be the dual operator of $A(\mu)$, which exists since $A(\mu)$ is densely defined (see