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THE JUMP CONDITIONS FOR SECOND ORDER QUASILINEAR DEGENERATE PARABOLIC EQUATIONS

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Abstract In this paper we consider the model problem for a second order quasilinear degenerate parabolic equation

$$\begin{cases} D_tG(u) = t^{2N-1}D_x^2K(u) + t^{N-1}D_xF(u) & \text{for} \quad x \in R, t > 0 \\ u(x,0) = A \quad \text{for} \quad x < 0, & u(x,0) = B \quad \text{for} \quad x > 0 \end{cases}$$

where $A < B$, and $N > 0$ are given constants; $K(u) \overset{\text{def}}{=} \int_A^u k(s)ds$, $G(u) \overset{\text{def}}{=} \int_A^u g(s)ds$, and $F(u) \overset{\text{def}}{=} \int_A^u f(s)ds$ are real-valued absolutely continuous functions defined on $[A,B]$ such that $K(u)$ is increasing, $G(u)$ strictly increasing, and $\frac{F(B)}{G(B)}G(u) - F(u)$ nonnegative on $[A,B]$. We show that the model problem has a unique discontinuous solution $u_0(x,t)$ when $k(s)$ possesses at least one interval of degeneracy in $[A,B]$ and that on each curve of discontinuity, $x = z_j(t) \overset{\text{def}}{=} s_jt^N$, where $s_j = \text{const.}$, $j = 1, 2, \cdots$, $u_0(x,t)$ must satisfy the following jump conditions:

1°. $u_0(z_j(t)-0,t) = a_j$, $u_0(z_j(t)+0,t) = b_j$, and $u_0(z_j(t),t) = [a_j, b_j]$ where $\{[a_j,b_j]; j = 1,2,\cdots\}$ is the collection of all intervals of degeneracy possessed by $k(s)$ in $[A,B]$, that is, $k(s) = 0$ a. e. on $[a_j,b_j]$, $j = 1,2,\cdots$, and $k(s) > 0$ a. e. in $[A,B] \setminus \bigcup_j [a_j,b_j]$, and

2°. $$\left(z_j'(t)G(u_0(x,t)) + t^{2N-1}D_xK(u_0(x,t)) + t^{N-1}F(u_0(x,t)) \right) \Big|_{x=z_j(t)-0}^{x=z_j(t)+0} = 0$$

Key Words Second order quasilinear degenerate parabolic equation; discontinuous solution; jump condition; two-point boundary value problem

Classifications 35K; 35D; 35B; 34B

1. Introduction

It has long been found that discontinuities may occur in a generalized solution to a second order quasilinear degenerate parabolic equation, just as they do in that to a first order quasilinear hyperbolic equation, which is itself a second order quasilinear degenerate parabolic equation where no second derivatives of the unknown function appear at all. However, only a few people really know why discontinuities occur in a generalized solution to a second order quasilinear degenerate parabolic equation and what jump conditions a generalized solution must satisfy on the set of its jump points; by a discontinuous solution we mean a generalized solution where discontinuities have already arisen.

As far as the uniqueness of a generalized solution of an initial or boundary value problem is concerned, it is indispensable that there are certain jump conditions which are satisfied by a generalized solution on the set of its jump points. As early as 1969, Vol'pert and Hudjaev[1] first put forward such jump conditions; it is a pity that the jump conditions were not completely correct yet. It was not until 1985 that Wu Zhuoqun[2] pointed out that one of the jump conditions was not true and gave it a correct form.

A perfect theory on generalized solutions to a second order quasilinear degenerate parabolic equation must give a round explanation of the reason why discontinuities occur in a generalized solution and of the relationship between discontinuities occurring in a deneralized solution and intervals of degeneracy possessed by the equation considered, and can provide some examples that exhibit an exact representation of a generalized solution and the set of its jump points. However, the jump conditions, in the absence of such explanation, were presented in both [1] and [2] only as consequences of a definition of generalized solutions to a second order quasilinear degenerate parabolic equation, while the definition came into being on the analogy of those to a first order quasilinear hyperbolic equation. In addition, there was no such example in [1] but only one in [2]. The example in [2], in my personal judgement, seems insufficient to expose the foregoing relationship. These were the reasons why the jump conditions presented in [1] had been amended sixteen years after they raised.

In this paper we shall in detail discuss the model problem for a second order quasilinear parabolic equation involving a small parameter $\varepsilon \geqslant 0$, of the form

$$D_t G(u) = t^{2N-1} D_x^2 (K(u) + \varepsilon u) + t^{N-1} D_x F(u) \quad \text{for} \quad x \in R, t > 0 \qquad (1)_\varepsilon$$

where D_t and D_x denote differentiation with respect to t and x, respectively, with the initial condition

$$u(x,0) = A \quad \text{for} \quad x < 0, \qquad u(x,0) = B \quad \text{for} \quad x > 0; \quad A < B \qquad (2)$$

Our aim is to naturally deduce the jump conditions $u_0(x,t)$ must satisfy on each of its curves of discontinuity. Here $u_0(x,t)$ is a discontinuous solution to the degenerate problem $(1)_0$-(2), namely, $u_0(x,t)$ is the pointwise limit of $u_\varepsilon(x,t)$, a solution to the nondegenerate problem $(1)_\varepsilon$-(2), as ε tends to zero.

2. Similarity Solutions

In this section we convert the problem of finding similarity solutions to the initial value problem $(1)_\varepsilon$-(2) into a two-point boundary value problem.

Throughout this paper we make the following two hypotheses:

H_1: $A < B$, and $N > 0$ are given constants;

H_2: $K(u) \overset{\text{def}}{=} \int_A^u k(s)ds$, $G(u) \overset{\text{def}}{=} \int_A^u g(s)ds$, and $F(u) \overset{\text{def}}{=} \int_A^u f(s)ds$ are real-valued absolutely continuous functions defined on $[A,B]$ such that $K(u)$ is increasing, $G(u)$ strictly increasing, and $\frac{F(B)}{G(B)} G(u) - F(u)$ nonnegative on