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HÖLDER ESTIMATE OF A QUASILINEAR PARABOLIC EQUATION WITH NONLINEAR OBLIQUE DERIVATIVE BOUNDARY CONDITION[①]

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The Hölder estimate of certain quasilinear parabolic equations with Dirichlet boundary condition has been studied in [1]. Now we extend it to the case of nonlinear oblique derivative boundary condition.

Let Ω be a bounded smooth domain in Euclidean n space R^n. Let positive constants $\lambda, \Lambda, T, \sigma, M$ satisfying $\lambda \leqslant \Lambda, \sigma > 2$. Denote $Q = \Omega \times (0, T]$. Let $u \in C^{2,1}(Q)$ satisfy

$$\mathscr{L}u = \sum a_{ij}(x,t,u,u_x)u_{x_ix_j} - u_t = b(x,t,u,u_x) \tag{1}$$

$$\lambda(1+|p|^{\sigma-2})|\xi|^2 \leqslant \sum a_{ij}\xi_i\xi_j \leqslant \Lambda(1+|p|^{\sigma-2})|\xi|^2, \forall\, \xi \in R^n \tag{2}$$

$$|b(x,t,u,p)| \leqslant \Lambda(1+|p|^\sigma) \tag{3}$$

$$\mathscr{M}u = \sum H_i(x,t,u,u_x)u_{x_i} = -H(x,t,u) \tag{4}$$

$$\lambda \leqslant H_n(x,t,u,p) \leqslant \Lambda \tag{5}$$

$$|H_i(x,t,u,p)| \leqslant \Lambda\ (1 \leqslant i \leqslant n), \quad |H(x,t,u)| \leqslant \Lambda \tag{6}$$

$$\max_Q |u| \leqslant M \tag{7}$$

The interior C^α estimate of u has been studied in [1]. Now discuss the C^α estimate near the lateral boundary. Suppose a part ω of $\partial\Omega$ lies on $x_n = 0$ and Ω lies entirely in the half space $x_n > 0$. Denote

$$d(P_1, P_2) = (|x^2 - x^1|^2 + |t^2 - t^1|)^{1/2}$$

where $P_1 = (x^1, t^1), P_2 = (x^2, t^2)$. Denote

$$d^*(P_1) = \min\{d(P_1, P) : P \in \{\partial\Omega \backslash \omega\} \times [0, T] \cup \Omega \times \{t = 0\}\}$$
$$d^*(P_1, P_2) = \min\{d^*(P_1), d^*(P_2)\}$$

Theorem *There exist constants* $\gamma(0 < \gamma < 1), C_1, C_2$ *depending on* $n, \lambda, \Lambda, \sigma, M$ *only such that when* $P_2 \in \omega \times [0, T], d(P_1, P_2) \leqslant C_1 d^*(P_1, P_2)$, *we have*

$$|u(P_2) - u(P_1)| \leqslant C_2 \left[\frac{d(P_1, P_2)}{d^*(P_1, P_2)^{1+\gamma}}\right]^\gamma \tag{8}$$

Proof Without loss of generality we can assume $H(x,t,u) \leqslant 0$, otherwise we re-

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place u by $u+\frac{\Lambda}{\lambda}x_n$. Assume that $P_2=(0,\cdots,0)$. Construct rectangles

$$K_j=\{(x,t):|x_i|<\xi^{-j}\equiv R_j(1\leqslant i\leqslant n-1),0<x_n<R_j,$$
$$-\tilde{\eta}\left(\frac{1}{2}\beta\eta^j\right)^{2-\sigma}(2nR_j)^\sigma\equiv -S_j<t<0\},j=1,2,\cdots$$

where $\xi=8n^{3/2}$, $3/4<\eta<1$, $\beta=2M\eta^{-j_0}$, $\tilde{\eta}$, η and j_0 are constants to be determined later. We shall prove by induction that

$$\underset{K_j}{\operatorname{osc}}\, u\leqslant\beta\eta^j\quad(j\geqslant j_0)\tag{9}$$

If the induction process is not valid, i. e.

$$\underset{K_{j-1}}{\operatorname{osc}}\, u\leqslant\beta\eta^{j-1}\tag{10}$$

$$\underset{K_j}{\operatorname{osc}}\, u>\beta\eta^j\tag{11}$$

We shall show that (11) leads to a contradiction.

Let

$$\max_{K_j} u=u(x^1,t^1)\equiv u_1,\ \min_{K_j} u=u(x^2,t^2)\equiv u_2$$

where $(x^1,t^1),(x^2,t^2)\in\overline{K}_j$. Denote $\frac{1}{2}(u_1+u_2)=u_0$,

$$K_j^1=\{(x,t):|x_i|<2nR_j\quad(1\leqslant i\leqslant n-1),$$
$$0<x_n<2nR_j,\quad -S_{j-1}<t<0\}$$

Without loss of generality we assume that

$$|K_j^1\cap\{u-u_0\leqslant 0\}|\geqslant\frac{1}{2}|K_j^1|$$

otherwise the above inequality is true when we substitute u by $-u$. Denote

$$K_j^2=\{(x,t):|x_i|<2nR_j\quad(1\leqslant i\leqslant n-1),$$
$$0<x_n<2nR_j,\quad -S_{j-1}<t<t^3\}$$

where $t^3\in[t^1-S_j^1,t^1]$. We have

$$|K_j^2\cap\{u-u_0\leqslant 0\}|\geqslant\frac{1}{2}|K_j^2|-\frac{2S_j}{S_{j-1}}|K_j^1|\geqslant\frac{1}{4}|K_j^2|$$

Let

$$\tilde{U}=\frac{\lambda}{2\Lambda}\left(e^{\frac{2\Lambda}{\lambda}(u-u_0)}-1\right)-\Lambda e^{\frac{4\Lambda}{\lambda}M}(S_{j-1}+t)$$

We prove that