J. Partial Differential Equations
Vol. 3, *No.* 3(1990), 54—62

THE HEAT KERNEL ON CONSTANT NEGATIVE CURVATURE SPACE FORM

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(Received Sept. 9, 1988; revised May 29, 1989)

Abstract Let M be a n-dimensional simply connected, complete Riemannian manifold with constant negative curvature. The heat kernel on M is denoted by $H_t^n(x,y)=H_t^n(r(x,y))$, where $r(x,y)=\mathrm{dist}(x,y)$.

We have the explicit formula of $H_t^n(x,y)$ for $n=2,3$, and the induction formula of $H_t^n(x,y)$ for $n\geqslant 4$ [1]. But the explicit formula is very complicated for $n\geqslant 4$. In this paper we give some simple and useful global estimates of $H_t^n(x,y)$, and apply these estimates to the problem of eigenvalue.

Key Words Constant negative curvature space form; heat kernel; eigenvalue.
Classification 58G.

1. The Estimates of $H_t^n(r)$

Suppose $k_M=-k^2$, $H_t^n(r)=H_t^n(r(x,y))$. It is well known that

$$H_t^2(r) = k(4\pi t)^{-3/2}e^{-k^2t/4} \cdot \int_r^{\infty} (\mathrm{ch}(k\rho) - \mathrm{ch}(kr))^{-1/2}e^{-\rho^2/(4t)}\rho d\rho \tag{1.1}$$

$$H_t^3(r) = (4\pi t)^{-3/2}e^{-k^2t}e^{-r^2/(4t)}\frac{kr}{\mathrm{sh}(kr)} \tag{1.2}$$

and

$$H_t^{n+2}(r) = -\frac{1}{2\pi}e^{-nk^2t}\frac{1}{k\mathrm{sh}(kr)}\frac{d}{dr}H_t^n(r) \tag{1.3}$$

So, the explicit formula for $n\geqslant 4$ is very complicated. It is difficult to get the global estimate of H_t^n from it, and it is more difficult to get the global lower bound.

In this section, we will give some natural, simple, and useful estimates of $H_t^n(r)$.

Theorem 1.1 *Let M be a n-dimensional simply connected complete Riemannian manifold with constant negative curvature $-k^2$, $H_t^n(r)=H_t^n(r(x,y))$ is the heat kernel on M. Then*

$$(4\pi t)^{-1}e^{-k^2t/3}e^{-r^2/(4t)}\left(\frac{r}{f(r)}\right)^{1/2} \leqslant H_t^2(r) \leqslant (4\pi t)^{-1}e^{-k^2t/4}e^{-r^2/(4t)}\left(\frac{r}{f(r)}\right)^{1/2}$$

$$H_t^3(r) = (4\pi t)^{-3/2}e^{-k^2t}e^{-r^2/(4t)}\left(\frac{r}{f(r)}\right)$$

$$c_n t^{-n/2}e^{-(n-1)^2k^2t/4}e^{-r^2/(4t)}\left(\frac{r}{f(r)}\right)^{(n-1)/2} \leqslant H_t^n(r)$$

$$\leqslant c_n t^{-n/2} e^{-(n-1)nk^2 t/6} e^{-r^2/(4t)} \left(\frac{r}{f(r)} \right)^{(n-1)/2}, \quad \textit{for } n \geqslant 4$$

where $c_n = (4\pi)^{-n/2}, f(r) = \frac{\mathrm{sh}(kr)}{k}$.

In order to prove this theorem, we first prove the following two lemmas.

Lemma 1. 1 *Suppose that* $f(r) = \mathrm{sh}(kr)$ *and*

$$s(r) = \left(\frac{1}{r} \right)^2 - \left(\frac{f'(r)}{f(r)} \right)^2 \tag{1.4}$$

Then $-k^2 \leqslant s(r) \leqslant -\frac{2}{3}k^2$.

Proof We first prove $s(r) \geqslant -k^2$. It suffices to prove

$$f^2(r) - r^2(f'(r))^2 + k^2 r^2 f^2(r) \geqslant 0$$

Set $F(r) = f^2(r) - r^2(f'(r))^2 + k^2 r^2 f^2(r)$, applying

$$f''(r) = k^2 f(r) \tag{1.5}$$

and computing directly, we can obtain

$$F(0) = 0, F'(0) = 0 \text{ and } F''(r) = 4k^2 f^2(r) \geqslant 0$$

Hence

$$F(r) \geqslant 0$$

Now, we prove $s(r) \leqslant -\frac{2}{3}k^2$. It suffices to prove

$$\left(\frac{\mathrm{ch}(x)}{\mathrm{sh}(x)} \right)^2 \geqslant \frac{2}{3} + \left(\frac{1}{x} \right)^2, \quad \text{for } x > 0$$

Set

$$G(x) = x^2 \mathrm{ch}^2(x) - \frac{2}{3} x^2 \mathrm{sh}^2(x) - \mathrm{sh}^2(x)$$

we have

$$G(0) = G'(0) = G''(0) = G'''(0) = 0$$

and

$$G^{(4)}(x) = \frac{32}{3} x \mathrm{sh}(2x) + \frac{8}{3} x^2 \mathrm{ch}(2x) \geqslant 0$$