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SYMMETRIES IN (1+1)-DIMENSIONAL INTEGRABLE SYSTEM WITH THEIR ALGEBRAIC STRUCTURES AND CONSERVED QUANTITIES①

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Abstract　In this paper we offer a general method of constructing symmetries and conserved quantities in the (1+1)-dimensional integrable system, prove the algebraic relations between symmetries, and what is more, give applications of this method in many integrable systems with physical significance.

Key Words　Integrable system; symmetry; conserved quantity; kac-moo algebra.
Classification　58G.

1.　Introduction

The concepts of symmetry and conserved quantity are very important in many physical theories. For a classical Hamiltonian system with $2N$ degrees of freedom, the existence of N conserved quantities in involution implies the complete integrability. These conserved quantities can be constructed from the symmetries of Lagrangian type of the system through the Noether theorem[1].

Many integrable Hamiltonian systems interesting physically have been found in the past two decades[2]. The motion equation of these systems can be solved in general by the inverse scattering transforms (called IST for short). On the other hand, they have infinite number of conserved quantites. As in the case of finite number of degrees of freedom, what associate with these conserved quantities are symmetries of Lagrangian type of these systems (called K symmetry for short)[4,5].

However it is found recently that except for K symmetry there still exist infinite number of other symmetries (called τ symmetry for short)[4,5]. This discovery is of profound significance because, for one thing, the symmetries K and τ form an infinite-dimensional Lie algebra and, for another, the hierarchies generated by these two symmetries, corresponding to the isospectral and nonisospectral cases respectively, can be solved by the IST[4,6,15].

The hierarchies generated by K symmetries, which are all motion equations of Hamiltonian systems, are well known and a great deal of discussion has been given to

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their symmetries[2]. But the hierarchies generated by τ symmetries (in the $(1+1)$-dimensional case) cannot be expressed as dynamical systems of Hamiltonian system in general. They depend explicitly on variables x and t and are often used to describe propagation of nonlinear wave in nonuniform media[7].

Discussion has never been given to the symmetries and conserved quantities of hierarchies generated by τ symmetries until recently the authors of this paper gave an effective method[4,8,9]. In this paper the method will be extended to a theory which can deal with, as is shown by many examples below, many problems concerning symmetries and conserved quantities of hierarchies generated by τ symmetries in the $(1+1)$-dimensional systems, continuous and discrete[15]. An analogous discussion to our paper may be referred in [16].

2. Symmetries and Their Algebraic Structures

Assume that M is a Schwartz space, the elements in which are C^∞ functional vectors defined on the real axis and vanish as fast as one wants for $|x|\to\infty$. S is a tangent bundle of M. S_u is the tangent space of $u\in M$ on M. For a field $K: M\to S$, i. e., $K(u)\in S_u$, $u\in M$, define its Gateaux derivative in the direction $\sigma(u)\in S_u$ as

$$K'(u)[\sigma]=\lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}K(u+\varepsilon\sigma) \tag{2.1}$$

This derivative is valid for any functional on M.

For an arbitrary $u\in M$, assume that there exist tangent vectors $K_0(u), \sigma_0(u)\in S_u$ (denoted by $K_1(u), \sigma_1(u)$ sometimes for convenience) and an operator $\Phi(u): S_u\to S_u$, and assume that they satisfy

1) Φ has hereditary, i, e, $\forall f, g\in S_u$, Φ satisfies

$$\Phi'[\Phi f]g-\Phi'[\Phi g]f=\Phi(\Phi'[f]g-\Phi'[g]f) \tag{2.2a}$$

or

$$(L_\Phi\Phi)g=(\Phi L_f\Phi)g \tag{2.2b}$$

Here, for $\forall f(u)\in S_u$, the definition of $L_f\Phi$ is [14]

$$L_f\Phi=\Phi'[f]-f'\Phi+\Phi f' \tag{2.3}$$

2) Φ is a strong symmetry of K_0, i. e.

$$L_{K_0}\Phi=0 \tag{2.4}$$

3) "The parameter α", i. e.

$$L_{\sigma_0}\Phi=\alpha \tag{2.5}$$

4) "The parameter β", i. e.

$$\begin{aligned}&[K_0,\sigma_0]=0, [K_0,\Phi\sigma_0]=\beta K_0\\&[\Phi K_0,\sigma_0]=\delta K_0, [\Phi K_0,\Phi\sigma_0]=\delta\Phi K_0\end{aligned} \tag{2.6}$$