J. Partial Differential Equations
Vol. 3, No. 4(1990), 31—48

THE VISCOSITY SPLITTING SOLUTIONS OF THE NAVIER—STOKES EQUATIONS

Ying Longan
(Peking University)

(Received August 21, 1989)

Abstract The solutions of the initial boundary value problems of the Navier-Stokes equations are constructed by means of a viscosity splitting scheme. Convergence results are proved.

Key Words Navier-Stokes equations; Euler equations; viscosity; split; convergence.
Classifications 35Q10; 76D05.

In [8]—[11] we studied a viscosity splitting scheme for the initial boundary value problems of the Navier—Stokes equations,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = \nu\triangle u + f, \text{in } \Omega \tag{0.1}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \tag{0.2}$$

$$u|_{x\in\partial\Omega} = 0 \tag{0.3}$$

$$u|_{t=0} = u_0 \tag{0.4}$$

where $x \in \Omega, f = f(x,t)$ and $u_0 = u_0(x)$ were given, $u = u(x,t) = (u_1, u_2, u_3)$ and $p = p(x,t)$ were unknowns, ν and ρ were positive constants. It was proved that if the original problem (0.1)—(0.4) admits a sufficiently smooth solution, then the approximate solutions converge to it. The bound of error was estimated. The background of this scheme is the computation of viscous incompressible flow with high Reynold's number.[2]

However, the necessary conditions for a sufficiently smooth solution include not only regularity but also some compatibility conditions of the prescribed data. For instance $\| u \|_3$ is unbounded near $t = 0$ unless there exists a solution p_0 of the overdetermined Neumann Problem[4]

$$\frac{1}{\rho}\triangle p_0 = \nabla \cdot (f(x,0) - (u_0 \cdot \nabla)u_0), \quad \text{in } \Omega$$

$$\frac{1}{\rho}\nabla p_0 = \nu\triangle u_0 + f(x,0) - (u_0 \cdot \nabla)u_0, \quad \text{on } \partial\Omega$$

Our purpose here is to get rid of these nonlocal conditions. The only compatibility condition in this paper is the condition of continuity, $u_0|_{x\in\partial\Omega}=0$. With some conditions on the regularity of the data u_0 and f, we will present a proof of convergence without any *a priori* hypothesis of existence, therefore our result also serves as a new proof of the local existence theorem. On the other hand, we will prove if there is a solution of the problem (0.1)—(0.4) on time interval $[0,T]$, then convergence holds on the same interval.

To fix our attention we consider three dimensional case here, our argument is also valid for the two dimensional problems. For simplicity let Ω be a bounded, simply connected domain in R^3 with sufficiently smooth boundary $\partial\Omega$.

Let us describe the scheme. We always denote by C a generic constant, we apply the usual notations $W^{m,p}(\Omega)$ and $\|\cdot\|_{m,p}$ for Sobolev spaces and the norms, and denote $|\cdot|_{m,p}$ the semi-norms. If $p=2$, then we write $H^m(\Omega)=W^{m,2}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$. Let Q be a projective operator

$$Q:(H^1(\Omega))^3 \to \{u\in (H_0^1(\Omega))^3;\nabla\cdot u=0\}$$

such that

$$\|Qu\|_s\leqslant C\|u\|_s,\quad \forall u\in (H^1(\Omega))^3\cap (H^s(\Omega))^3, \forall s>1/2$$

The following scheme for approximate solutions is essentially the same as that in [8]—[11]: Let $k>0$ be the length of time step. If $\tilde{u}_k(ik-0), i=0,1,\cdots$, is known, then we solve $u_k(t)$ and $\tilde{u}_k(t)$ on $[ik,(i+1)k)$ in two steps. Firstly u_k satisfies

$$\frac{\partial u_k}{\partial t}+\frac{1}{\rho}\nabla p_k=\nu\triangle u_k+f+\frac{1}{k}(I-Q)\tilde{u}_k(ik-0) \tag{0.5}$$

$$\nabla\cdot u_k=0 \tag{0.6}$$

$$u_k|_{x\in\partial\Omega}=0 \tag{0.7}$$

$$u_k(ik)=Q\tilde{u}_k(ik-0) \tag{0.8}$$

Secondly, $\tilde{u}$ satisfies

$$\frac{\partial\tilde{u}_k}{\partial t}+(\tilde{u}_k\cdot\nabla)\tilde{u}_k+\frac{1}{\rho}\nabla\tilde{p}_k=0 \tag{0.9}$$

$$\nabla\cdot\tilde{u}_k=0 \tag{0.10}$$

$$\tilde{u}_k\cdot n|_{x\in\partial\Omega}=0 \tag{0.11}$$

$$\tilde{u}_k(ik)=u_k((i+1)k-0) \tag{0.12}$$

Then we repeat the above procedure. Here n is the unit outward normal on $\partial\Omega$, I is the identity operator, and $\tilde{u}_k(-0)$ is u_0. An example of the operator Q will be given in Section 1.