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HIROTA-TYPE EQUATIONS, SOLITON SOLUTIONS, BÄCKLUND TRANSFORMATIONS AND CONSERVATION LAWS

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Abstract In this paper, first a class of Hirota-type equations

$$\sum_{i=1}^{l} H_i(D_x, D_t, D_y)[F_i(D_x, D_t, D_y)f \cdot f] \cdot [G_i(D_x, D_t, D_y)f \cdot f] = 0$$

are considered. By imposing certain conditions on F_i, G_i and H_i, we show that the above-mentioned equation possesses one-soliton solution. Secondly we present a new integrable equation which is an extention of Novikov—Veselov equation and Ito equation. We obtain a Bäcklund transformation (BT) of this equation. Finally we consider a generalized equation of Ramani and Sawada-Kotera, and obtain its BT. Starting with the BT an infinite number of conservation laws are derived.

Key Words Soliton; Bäcklund transformation; Conservation law.
Classification 35Q20.

1. Introduction

It is known that many important nonlinear integrable equations such as KdV, Boussinesq, KP can be transformed into the following bilinear equation

$$F(D_x, D_t, D_y)f \cdot f = 0 \tag{1}$$

where F is an even degree polynomial and the bilinear operator $D_x^l D_t^m D_y^n$ is defined by

$$D_x^l D_t^m D_y^n a(x,t,y) \cdot b(x,t,y)$$
$$\equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n a(x,t,y)b(x',t',y')\Big|_{\substack{x'=x\\ t'=t\\ y'=y}}$$

Once a nonlinear evolution equation is transformed into a bilinear equation, the latter seems to be easier to treat. Hirota proved [1] that there exist at least 2-soliton solutions for (1) if $F(0,0,0)=0$. Moreover a sufficient condition on the existence of N-soliton solutions for (1) was also given. Very recently Hietarinta [2] investigated the condition on the existence of 3-soliton solutions for (1) and found some new differential equations possessing 3-soliton solutions. However, the pity is that equation (1) is only a very small class of nonlinear evolution equations. Therefore it is necessary to generalize (1) and then proceed to find more and more new integrable equations.

This paper is organized as follows. In Section 2, we consider a generalized form of Hirota-type equation (1)

$$\sum_{k=1}^{l} H_k(D_x,D_t,D_y)[F_k(D_x,D_t,D_y)f\cdot f]\cdot[G_k(D_x,D_t,D_y)f\cdot f]=0 \qquad (2)$$

where F_k, G_k, H_k are all constant coefficient polynomials of D_x, D_t, D_y. By imposing certain conditions on F_k, G_k and H_k, we show that there exists at least an one-soliton solution to (2). In Section 3, we give a new Hirota-type integrable equation which is an extension of Novikov-Veselov equation [3],[4] and Ito equation [5]. A Bäcklund transformation for this new equation is obtained. At the same time we also consider a generalized equation of Ramani [2] and Sawada-Kotera [6], and obtain a BT. Starting with the BT, an infinite number of conservation laws are derived. In addition, we give an appendix which lists some bilinear operator identities used in the paper.

2. Soliton Solutions to Equation (2)

In this section, we are going to search for soliton solutions to (2). To this end we assume that F_k, G_k, H_k of (2) satisfy the following two conditions:

1) Both F_k and G_k are even degree polynomials and $H_k(k=1,2\cdots l)$ are either all even degree polynomials or all odd degree polynomials.

2) $F_k(0,0,0)=H_k(0,0,0)=0, k=1,2,\cdots l.$

Let us expand f formally in powers of an arbitrary parameter ε as

$$f=\sum_{n=0}^{\infty} f_n\varepsilon^n, \quad f_0=1$$

Substituting this expansion into (2) and comparing the coefficients $\varepsilon^m(m=1,2,\cdots)$, we obtain

$$\sum_{k=1}^{l} H_k(D_x,D_t,D_y)\sum_{i=0}^{m}[F_k(D_x,D_t,D_y)(\sum_{j=0}^{m-i} f_{m-i-j}\cdot f_j)]\cdot$$

$$\cdot[G_k(D_x,D_t,D_y)(\sum_{j=0}^{i} f_{i-j}\cdot f_j)]=0 \qquad (3)$$

By use of hypotheses 1), 2) and formulae (A · 1), (A · 2) in the appendix, it follows from (3) that

$$\begin{aligned}0=&\sum_{k=1}^{l} H_k(D_x,D_t,D_y)[F_k(D_x,D_t,D_y)f_0\cdot f_0]\cdot[G_k(D_x,D_t,D_y)(\sum_{j=0}^{m} f_{m-j}\cdot f_j)]\\&+\sum_{k=1}^{l} H_k(D_x,D_t,D_y)[F_k(D_x,D_t,D_y)(\sum_{j=0}^{m} f_{m-j}\cdot f_j)]\cdot[G_k(D_x,D_t,D_y)f_0\cdot f_0]\\&+\sum_{k=1}^{l} H_k(D_x,D_t,D_y)\sum_{i=1}^{m-1}[F_k(D_x,D_t,D_y)(\sum_{j=0}^{m-i} f_{m-i-j}\cdot f_j)]\\&\cdot[G_k(D_x,D_t,D_y)(\sum_{j=0}^{i} f_{i-j}\cdot f_j)]\\=&\sum_{k=1}^{l} H_k(D_x,D_t,D_y)[F_k(D_x,D_t,D_y)(f_m\cdot f_0+f_0\cdot f_m)]\cdot G_k(0,0,0)\end{aligned}$$