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THE CAUCHY PROBLEM FOR A SPECIAL SYSTEM OF QUASILINEAR EQUATIONS

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Abstrart We have obtained in this paper the existence of weak solutions to the Cauchy problem for a special system of quasilinear equations with physical interest of the form

$$\begin{cases} \frac{\partial}{\partial t}(u+qz)+\frac{\partial}{\partial x}f(u)=0 \\ \frac{\partial z}{\partial t}+k\varphi(u)z=0 \end{cases}$$

for the assumed smooth function $\varphi(u)$ by employing the viscosity method and the theory of compensated compactness.

Key Words Entropy pair; weak solution.
Classification 35L.

1. Introduction

A physical model of combustion reads

$$\begin{cases} \frac{\partial}{\partial t}(u(x,t)+qz(x,t))+\frac{\partial}{\partial x}f(u(x,t))=0 \\ \frac{\partial}{\partial t}z(x,t)+k\varphi(u(x,t))z(x,t)=0, \quad (x,t)\in R_+^2 \end{cases} \tag{1.1}$$

where u denotes a lumped variable representing some features of density, velocity and temperature, z represents the density of unburn fraction in fluid while k is the rate of chemical reaction and q is specific binding energy, both of them are positive constants. $f(u)$ is a smooth function in R and $\varphi(u)=1$ for $u\geqslant 0$ and $\varphi(u)=0$ for $u<0$. This model was once mentioned by Majda [1]. Teng & Ying have widely investigated this problem; in particular, the existence and uniqueness of the solution on the Riemann problem for (1.1) has been obtained on condition that $f(u)$ is strongly convex for $u>0$ and $f''>0$ for $u\leqslant 0$ [2,3]. They also established the existence of generalized solutions when $k=+\infty$ under more restrictions on $f(u)$ by the difference scheme [4,5]. We use, here, the viscosity method and the theory of compensated compactness to achieve the existence of global weak solutions of the Cauchy problem for (1.1) for smooth $\varphi(u)$ when $f''(u) \neq 0$ a. e. in R. Note that (1. 1) reduces to

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) - kq\varphi(u(x,t))z(x,t) = 0 \\ \frac{\partial}{\partial t}z(x,t) + k\varphi(u(x,t))z(x,t) = 0, \quad (x,t) \in R_+^2 \end{cases} \tag{1. 2}$$

It is easy to see that the weak solutions of Cauchy problem for (1. 1) are equivalent to those for (1. 2). Thus we only pay our attentions to (1. 2) with the initial values

$$(u(x,0),z(x,0)) = (u_0(x),z_0(x)), \quad x \in R \tag{1. 3}$$

where $u_0(x), z_0(x)$ are bounded and measurable in R. The programme is as follows: firstly we shall establish the existence and a priori estimate of the global smooth solution $(u^\varepsilon(x,t),z^\varepsilon(x,t))$ for the following parabolic equations

$$\begin{cases} u_t^\varepsilon(x,t) + f(u^\varepsilon(x,t))_x - kq\varphi(u^\varepsilon(x,t))z^\varepsilon(x,t) = \varepsilon u_{xx}^\varepsilon(x,t) \\ z_t^\varepsilon(x,t) + k\varphi(u^\varepsilon(x,t))z^\varepsilon(x,t) = \varepsilon z_{xx}^\varepsilon(x,t), \quad \varepsilon > 0, (x,t) \in R_+^2 \end{cases} \tag{1. 4}$$

with the initial values

$$(u^\varepsilon(x,t),z^\varepsilon(x,t))|_{t=0} = (u^\varepsilon(x,0),z^\varepsilon(x,0)), \quad x \in R \tag{1. 5}$$

here $u^\varepsilon(x,0), z^\varepsilon(x,0)$ are step functions which are constants $u_n^\varepsilon, z_n^\varepsilon$ in the interval $n\varepsilon \leqslant x < (n+1)\varepsilon, n \in Z$ and converge to $u_0(x), z_0(x)$ almost everywhere in R, respectively; secondly we shall find out the subsequences of smooth functions $\{u^\varepsilon(x,t)\}, \{z^\varepsilon(x,t)\}$ such that the subsequence of $\{u^\varepsilon(x,t)\}$ converges in the sense of strong topology to a function $u(x,t)$ and the subsequence of $\{z^\varepsilon(x,t)\}$ converges in the sense of weak-star topology to a function $z(x,t)$. Finally we shall show the function pair $(u(x,t), z(x,t))$ is just the weak solution of (1. 2) and (1. 3).

2. Global Smooth Solutions

To reach the existence of the global smooth solution to (1. 4) and (1. 5) we investigate the following integral equations (for simplicity we omit ε s of $u^\varepsilon(x,t)$ and $z^\varepsilon(x,t)$)

$$\begin{cases} u(x,t) = \int_{-\infty}^{\infty} u(\xi,0)G(x,t;\xi,0)d\xi + \int_0^t d\tau \int_{-\infty}^{\infty} [f(u(\xi,\tau)) \frac{\partial G}{\partial \xi}(x,t;\xi,\tau) \\ \qquad + kq\varphi(u(\xi,\tau))z(\xi,\tau)G(x,t;\xi,\tau)]d\xi \\ z(x,t) = \int_{-\infty}^{\infty} z(\xi,0)G(x,t;\xi,0)d\xi - k\int_0^t d\tau \int_{-\infty}^{\infty} \varphi(u(\xi,\tau)) \\ \qquad \times z(\xi,\tau)G(x,t;\xi,\tau)d\xi \end{cases} \tag{2. 1}$$

where $G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi\varepsilon(t-\tau)}} \exp\left\{-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}\right\}$.