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THE RELATION BETWEEN THE DIFFERENTIABILITY OF SOLUTION AND LOWER-ORDER TERMS OF THE CAUCHY PROBLEM FOR A CLASS OF WEAKLY HYPERBOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

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Abstract This paper discusses a class of weakly hyperbolic equations with singular coefficients. We first set up the energy inequality, and then discuss the wellposedness of the Cauchy problem by means of the energy inequality, and the relation between the differentiability of solution and lower-order terms.

Key Words Differentiability of solution; Lower-order terms; Cauchy problem; Energy inequality.

Classification 35L80.

1. Introduction

Much study has been made to the relation between the differentiability of solution and lower-order terms of the Cauchy problem for weakly hyperbolic equations. V. Y. Ivrii and V. M. Petkof have shown in [1] that the Cauchy problem for a linear partial differential operator is L_2-wellposed with a loss of one derivative, with a necessary and sufficient condition that the operator is strictly hyperbolic. M. Zeman pointed out in [2] that if a linear hyperbolic operator has smooth double characteristics, its Cauchy problem is L_2-wellposed and the solution has a loss of at most two derivatives. T. Mandai proved in [3] that the differentiability of solution of the Cauchy problem for weakly hyperbolic operator with constant multiplicity characteristics is determined wholly by the multiplicity of its characteristics. These results show that for the operators mentioned-above, the loss of the differentiability of solution is determined wholly by the multiplicity of characteristics. But for general weakly hyperbolic operators, this conclusion will not remain true. The problem studied by Qi Minyou in [4] is exactly a powerful example in this case. T. Mandai studied in [3] the relation between the differentiability of solution and lower-order terms for general hyperbolic operators with energy inequality holding. This paper discusses a class of weakly hyperbolic equations with singular coefficients. We first set up the energy inequality, and then discuss the well-posedness of the Cauchy problem by means of the energy inequality, and the relation between the differentiability of solution and lower-order terms. Our results show that for the operator discussed here, the differentiability of solution is determined wholly by the norm of the lower-order operator with singular coefficients.

2. Notations and Definitions

We are concerned with the following equation

$$(\frac{\partial}{\partial t} - it^{\rho}a(x,t;D_x))(\frac{\partial}{\partial t} + it^{\rho}a(x,t;D_x))u + \frac{\beta(x,t;D_x)}{t}\frac{\partial u}{\partial t} = f(x,t) \qquad (1)$$

where $x \in R^n, D_x = (\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}), a(x,t;D_x)$ is a first order pseudo-differential operator with respect to x. t is regarded as a parameter with a real symbol $a_A = a(x,t;\xi)$, which is linearly homogeneous with respect to ξ and when $\xi \neq 0, a(x,t;\xi)$ is nonzero. $\beta(x,t;D_x)$ is an operator of order zero, $f(x,t)$ is a decently smooth fynction of x and t, and ρ is a positive constant.

We shall consider the homogeneous Cauchy problem for equation (1), i. e. , $\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0} = 0$, but the number of the initial conditions relates to β.

Definition 1 *We say that the Cauchy problem of a partial differential operator P of order m is well-posed in the sense of Hadamard, if it exists a unique solution which depends continuously on initial data and the function f $(Pu = f)$.*

Definition 2 *We say that the Cauchy problem of a partial differential operator P of order m is L_2-wellposed with the loss of r times derivatives, if it is well-posed in the sense of Hadamard and there is the following energy inequality*

$$\sum_{|\alpha| \leq m-r} \| D^{\alpha}u \|_s \leq C \| Pu \|_s \qquad (2)$$

where $\| v(\cdot, t) \|_s$ stands for the norm equipped to the Sobolev space H_s.

Denoting $Pu = f$, (2) shows that if $f \in H_s$, then $u \in H_{s+m-r}$.

In the above definitions the loss of t-derivative and that of x-derivative are not distinguished. Since the variable t is in a more special position in our discussion, it is necessary to distinguish the loss of the x-differentiability of solution u from that of the t-differentiability. For this purpose we shall introduce the notation

$$\| u(\tau) \|_{q,s} = \left(\sum_{\substack{|\alpha| \leq q \\ j \leq s}} \int | D_x^{\alpha} D_t^j u(\tau) |^2 dx \right)^{1/2} \qquad (3)$$

Definition 3 *We say that the solution u of the Cauchy problem of a partial differential op-*