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ON THE NATURAL GROWTH QUASILINEAR ELLIPTIC EULER EQUATIONS①

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Abstract In this paper, we consider the eigenvalue problem and the Dirichlet problem of general Euler equations under the natural growth condition.

Key Words Natural growth; Elliptic Euler equation; Eigenvalue problem; Variational method.

Classifications 35J65; 49A40.

1. Preface and Assumptions

In this paper the eigenvalue of general Euler equations under the natural growth condition is first discussed:

$$\begin{cases} -\dfrac{d}{dx_i}F_i(x,u,Du)+F_u(x,u,Du)=\lambda|u|^{p-2}u, & x\in\Omega \\ u(x)-\omega(x)\in W_0^{1,m}(\Omega) \end{cases} \tag{1}$$

where

$$F_i=\frac{\partial}{\partial q_i}F(x,u,q), q=(q_1,\cdots,q_n), F_u=\frac{\partial}{\partial u}F(x,u,q)$$

Ω is a bounded domain in R^n, $n>m$, $m\leqslant p<\dfrac{nm}{n-m}$, $\omega(x)\in W^{1,m}(\Omega)\cap L_\infty(\partial\Omega)$. Both $W_0^{1,m}(\Omega)$ and $W^{1,m}(\Omega)$ are Sobolev spaces.

The special case of this problem, i. e. $F(x,u,q)=a_{ij}(x,u)q_iq_j+c(x)u^2$ and $\omega(x)=0$, with the assumptions:

$$a|\xi|^2\leqslant a_{ij}(x,u)\xi_i\xi_j\leqslant a_1|\xi|^2, \quad a>0$$
$$-\frac{1}{2}u\partial_u a_{ij}(x,u)\xi_i\xi_j\leqslant \alpha a_{ij}(x,u)\xi_i\xi_j, \quad 0<\alpha<1, \partial_u=\frac{\partial}{\partial u}$$

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$$\lim_{u\to\infty} u\partial_u a_{ij}(x,u) = 0 \tag{2}$$

has been discussed in [1].

We consider multiple integrals of the form $I(u) = \int_\Omega F(x,u,Du)dx$, then we know that the variational problem:

$$I(u) = \inf_{v\in K} I(v)$$
$$K = \{u \mid u \in W^{1,m}(\Omega) \text{ and } u - \omega \in W_0^{1,m}(\Omega)\}$$

under some sufficient conditions relevant to $F(x,u,q)$, has its solution (see [2]). A similar method can be used to prove the existence of the solution to the variational problem:

$$I(u) = \inf_{v\in E} I(v), \quad E = \{u \mid u \in K, \| u \|_r = 1\} \tag{3}$$

here $\| u \|_r = \| u \|_{L_r}$.

However, when $F(x,u,q)$ grows naturally, $I(u)$ could be differentiable only when $u \in L_\infty(\Omega)$ (see [3]). Unfortunately, it is quite difficult to verify that the solution $u(x)$ to variational problem (3) is bounded, because $F(x,u,q)$ grows naturally, in addition, the problem is restricted on E.

In this paper this difficulty will be overcome. Shortly speaking, if $u(x)$ is the solution to (3), we firstly prove, for some special test functions φ, $\varphi(x) = \mathrm{sign}u \cdot \max(|u|-k,0)$, $I((u+t\varphi)/\| u+t\varphi \|_r)$ is differentiable about t, when $t=0$. Then, we derive:

$$\int_\Omega [F_i(x,u,Du)D_i\varphi + F_u(x,u,Du)\varphi]dx = \lambda\int_\Omega |u|^{r-2}u\varphi dx \tag{4}$$

Next this fact shows the boundedness of u, thus we learn (4) is satisfied for all $\varphi \in W_0^{1,m}(\Omega)$. The solution of the variational problem (3), consequently, is the weak solution of the problem (1). Here we can omit the condition (2), and we can also consider non-homogeneous Dirichlet problem, apart from these, $a_{ij}(x,u)$ need not to be uniformly bounded about u. We may apply our method to discuss the eigenvalue problem of general quasilinear Euler equations.

Because (2) is omitted, the proof of (4) about some special test functions φ becomes rather complex and different from the other paper. Nevertheless, we easily apply the properties of Giorgi functions in [2] to prove $u \in L_\infty(\Omega)$ finally.

Surely, we can also use this method to discuss Dirichlet problem of general quasilinear Euler equations.

Some assumptions about $F(x,u,q)$:

(i) Suppose $F(x,u,q)$ is measurable about x, and is continuously differentiable