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CLASSICAL SOLUTION TO THE ELECTROPAINTING PROBLEM

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Abstract The mathematical modelling of the electrodeposition phenomenon leads to a linear elliptic partial differential equation subject to nonlinear evolutionary mixed boundary conditions. In this paper, the existence, uniqueness and regularity of classical solution are proved for the electropainting problem when "dissolution current" is zero.

Key Words electropainting problem; classical solution.
Classification 35R35.

1. Introduction

We consider a time-dependent elliptic free boundary problem associated with an electropaint process. The problem is to find a pair $(v(x,t), h(x,t))$ such that in an annular region $\Omega \subset R^N (N \geqslant 2)$ with outer boundary S and inner boundary Γ there hold

$$\triangle v = 0 \quad \text{in } \Omega \tag{1.1}$$
$$v = 1 \quad \text{on } S \tag{1.2}$$
$$hv_n = v \quad \text{on } \Gamma \tag{1.3}$$
$$h_t = (v_n - \varepsilon)^+ \quad \text{on } \Gamma \tag{1.4}$$
$$h(x,0) = 0 \quad \text{on } \Gamma \tag{1.5}$$

where v_n is the inward normal derivative on Γ, $(z)^+ = \max(0,z)$, $\varepsilon > 0$ is a given constant.

Similar problems were considered by Hansen and McGeough ([1]), Aitchison, Lacey and Shillor ([2]), Caffarelli and Friedman ([3]), and Márquez and Shillor ([4]). The first two dealt with the modelling aspects of the electropaint process and numerical experiments. The latter two dealt with the time-discretized version of (1.1)–(1.5) and the electropainting problem with overpotentials respectively.

A problem of the type (1.1)–(1.5) can be considered as a model for the following process (see [1] or [2]):

A metal body with an outer surface Γ, to be painted, is immersed in a tank with an electrolytic solution. The solution occupies the region Ω such that $\partial\Omega = \Gamma \cup S$, where S is the inner surface of the tank. The metal part, which is usually called "the work piece", is connected to an electric potential source, the tank itself (S) serves as the other electrode and as a result of the flow of the electric current in the solution and into Γ, the process of paint deposition takes place on Γ. The unknown function v stands for the electric potential and in the boundary conditions on Γ, the unknown function h is the thickness of the paint coat. The existence of a "dissolution current" $\varepsilon>0$, that was postulated in [2], assures that there is paint deposition only at those points of Γ where the current v_n satisfies $v_n>\varepsilon$. When the dissolution current can be neglected, the model becomes

$$\triangle v=0 \quad \text{in } \Omega \tag{1.6}$$

$$v=1 \quad \text{on } S \tag{1.7}$$

$$hv_n=v \quad \text{on } \Gamma \tag{1.8}$$

$$h_t=v_n>0 \quad \text{on } \Gamma \tag{1.9}$$

$$h(x,0)=0 \quad \text{on } \Gamma \tag{1.10}$$

and we refer to (1.6)—(1.10) as problem (P). Hansen and McGeough first proposed this model in [1], and in addition, they also described two major features of the process, the "saturation effect" and the "levelling effect", via numerical experiment.

In all these papers mentioned above, however, it seems that no classical solution has been obtained for the electropainting problem (1.1)—(1.5) or (P).

In the present paper, we shall study the existence, uniqueness and regularity of classical solution to the problem (P)

Definition 1.1 *A classical solution to the problem (P) is a pair $(v(x,t),h(x,t))$ of functions such that*

(i) $v(x,t)\in C^0(\overline{\Omega}\times[0,\infty))$

$v(\cdot,t)\in C^1(\overline{\Omega})\cap C^2(\Omega) \quad \forall\ t\geq 0$

$h(x,t)\in C^0(\Gamma\times[0,\infty))$

$h_t(x,t)\in C^0(\Gamma\times(0,\infty))$

(ii) $\dfrac{\partial}{\partial n}\displaystyle\int_0^t v(x,\tau)d\tau = h(x,t) \quad$ *on* $\Gamma\times(0,\infty)$

(iii) (1.6)—(1.10) *are satisfied for any* $t>0$.

Our main results are the existence theorem and the regularity theorem stated as follows:

Theorem 1.1 *Let $\partial\Omega\in C^4$. Then there exists a unique classical solution of the problem (P).*

Theorem 1.2 *Let $\partial\Omega\in C^{k+\alpha}(k\geq 4, 0<\alpha<1)$, and $v(x,t)$ be the classical solution of the problem (P). Then*

$$v(x,t)\in C^{k+\alpha}(\overline{\Omega}\times(0,\infty))$$

and moreover

$$D_t^m v(x,t)\in C^{k+\alpha}(\overline{\Omega}\times(0,\infty)),\quad m=1,2,\cdots$$