J. Partial differential Equations
Vol. 4 No. 1(1991), 77—96

NONTRIVIAL SOLUTIONS FOR SOME SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL SOBOLEV EXPONENTS①

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(Received July 9, 1989; revised June 4, 1990)

Abstract Let Ω be a bounded domain in $R^n(n\geqslant 4)$ with smooth boundary $\partial\Omega$. We discuss the existence of nontrivial solutions of the Dirichlet problem

$$\begin{cases} -\triangle u = a(x)|u|^{4/(n-2)}u + \lambda u + g(x,u), & x\in\Omega \\ u=0, \quad x\in\partial\Omega \end{cases}$$

where $a(x)$ is a smooth function which is nonnegative on $\overline{\Omega}$ and positive somewhere, $\lambda>0$ and $\lambda\notin\sigma(-\triangle)$. We weaken the conditions on $a(x)$ that are generally assumed in other papers dealing with this problem.

Key Words Semilinear elliptic equation; Sobolev exponent; Critical value; Critical point; (P · S) condition.

Classifications 35J20; 35J60; 35D05.

1. Introduction

Let $\Omega\subset R^n(n\geqslant 4)$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we are concerned with the problem of finding u satisfying the following semilinear elliptic problem

$$(P1)\qquad \begin{cases} -\triangle u = a(x)|u|^{4/(n-2)}u + \lambda u + g(x,u), & x\in\Omega \\ u=0, \quad x\in\partial\Omega \end{cases}$$

where λ is a positive constant, $a(x)$ is a smooth function on Ω which is nonnegative and positive somewhere, $g(x,u)$ is a lower—order perturbation of $|u|^{(n+2)/(n-2)}$ in the sence that

$$\lim_{u\to\infty}\frac{g(x,u)}{|u|^{(n+2)/(n-2)}}=0 \text{ and } g(x,0)=0$$

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① The project supported by Natural Science Foundation of Zhejiang Province.

The important results concerning the problem (P1) have been obtained by H. Brezis and L. Nirenberg [1], they showed that (P1) possesses a positive solution for $a(x)=1, 0<\lambda<\lambda_1$ and $g(x,u)=0$. There has been some progress in this direction due to D. Fortunato, A. Capozzi and G. Palmieri [2], J. F. Escobar[3], C. F. Wang and R. Y. Xue [4], W. D. Lu and C. J. He [5]. In [3] the author showed that, for $0<\lambda<\lambda_1, g(x,u)=0$ and $a(x)$ satisfying some technical restriction, (P1) possesses a positive solution. For $a(x)\geqslant\delta>0$ (δ is a positive constant) and $g(x,u)$ satisfying other conditions, W. D. Lu and C. J. He [5] have proved that there is a constant λ_j^* such that (P1) has at least one nontrivial solution for any $\lambda\in(\lambda_j^*,\lambda_j)(j=1,2,\cdots)$. In this paper, we follow the method developed by H. Brezis and L. Nirenberg [1], weaken the conditions on $a(x)$ that are generally assumed in other papers dealing with these problems with critical Sobolev exponents, extend the results in [1], [2] and [3].

2. Some Preliminaries

Define

$$I(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx-\frac{\lambda}{2}\int_\Omega u^2dx$$

$$-\frac{n-2}{2n}\int_\Omega a(x)|u|^{2n/(n-2)}dx-\int_\Omega G(x,u)dx,\quad u\in H_0^1(\Omega)\qquad(2.1)$$

$$G(x,u)=\int_0^u g(x,u)du$$

It is well known that the solutions of (P1) correspond to critical points of the functional $I(u)$.

Let $\|\cdot\|$, $|\cdot|_p$ denote respectively the norms in $H_0^1(\Omega)$ and $L^p(\Omega)$ $(1\leqslant p<+\infty)$ and let

$$S=\inf\{\|u\|^2: |u|^2_{2n/(n-2)}=1,\quad u\in H_0^1(\Omega)\}$$

denote the best constant for the embedding $H_0^1(\Omega)\hookrightarrow L^{2n/(n-2)}(\Omega)$. We denote by $0<\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\cdots$ the sequence of eigenvalues of the eigenvalue problem

$$\begin{cases}-\triangle u=\lambda u, & \text{in }\Omega\\ u=0, & \text{on }\partial\Omega\end{cases}$$

and $\lambda_0=0$.

We are now in a position to collect the various hypotheses to be placed on the non-