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EXISTENCE AND NONEXISTENCE OF GLOBAL SMOOTH SOLUTIONS FOR DAMPED p-SYSTEM WITH " REALLY LARGE" INITIAL DATA*

Lin Longwei   Yang Tong
(Dept. of Math., Zhongshan Univ.)

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Abstract In this paper, we extend the results in [4] to general $p(v)$, and remove the condition (A) for existence, but keep it for nonexistence.

Key Words quasilinear hyperbolic systems; global smooth solutions; existence and nonexistence.

Classification 35L65.

Consider the initial value problem for damped p-system

$$v_t - u_x = 0, \quad u_t + p(v)_x = -2Au, \quad A > 0 \tag{E$_1$}$$

$$v(0,x) = v_0(x), \quad u(0,x) = u_0(x) \tag{I$_1$}$$

where $p(v) \in C^2(b,\infty)$, $p'(v) < 0$, $p''(v) > 0$, b is a finite number or $-\infty$. The characteristics of $(\mathrm{E})_1$ are $\lambda = -\sqrt{-p'(v)}$, $\mu = \sqrt{-p'(v)}$. The Riemann invariants are taken as $r = u + \Phi(v)$, $s = u - \Phi(v)$, where $\Phi = \int^v \mu(s)ds$. The Riemann invariants give a one to one smooth mapping from $\Omega_1 = \{(u,v)|u \in R, b < v < \infty\}$ onto $\Omega_2\{(r,s)|2\Phi(b) < r - s < 2\Phi(\infty)\}$; and transform $(\mathrm{E})_1$, $(\mathrm{I})_1$ into

$$r_t + \lambda r_x = -A(r+s),\ s_t + \mu s_x = -A(r+s),\ t > 0, x \in R \tag{E}$$

$$r(0,x) = r_0(x),\ s(0,x) = s_0(x),\ x \in R \tag{I}$$

Nishida [1], Slemrod [2, 3] studied the existence and nonexistence of global smooth solutions with "small" data. Lin and Zheng [4] studied that with "large" initial data under the following condition (A), where $p(v) = k^2 v^{-\gamma}, 0 < \gamma < 3$.

$$\left\{(r,s)\,\Big|\inf_x r_0(x) \le r \le \sup_x r_0(x), \inf_x s_0(x) \le s \le \sup_x s_0(x)\right\} \subset \Omega \tag{A}$$

where $\Omega = \{(r,s)|2\Phi(b) < r - s < 2\Phi(\infty)\}$.

We now extend the results in [4] to general $p(v)$. But we remove (A) for existence, and keep it for nonexistence. We conjecture that (A) can also be removed for nonexistence under the following condition (C). Under condition (A) the initial data are not really large.

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Note When $A = 0$, the nonexistence theorem has been proven under condition (A) only (cf. [5], [6]).

In this paper, we always suppose that the initial data satisfy the following condition

$$u_0(x),\ v_0(x) \geq \delta_0 > 0 \in C^1(R), |u_0(x)| + |v_0(x)| + |u_0'(x)| + |v_0'(x)| \leq M \qquad \text{(C)}$$

where M can be arbitrarily large. Moreover, we always suppose that $p(v) \in C^3(b, \infty)$, $\dot{p}(v) < 0$, $\ddot{p}(v) > 0$, $\lim_{v \to b+} p(v) = \infty$, where b is a finite number or $-\infty$, and the super-dot "$\cdot$" denotes a differentiator with respect to v. Obviously, without loss of generality, we can assume $b = 0$ as b is finite.

Lemma 1 *If* $2\dddot{p}\,\dot{p} - 3\ddot{p}^2 \geq 0$, *then b is a finite number.*

Proof Since $2\dddot{p}\,\dot{p} - 3\ddot{p}^2 \geq 0$, $\dot{p} < 0$, $\ddot{p} > 0$, we have

$$2\frac{\dddot{p}}{\ddot{p}} \leq 3\frac{\ddot{p}}{\dot{p}} \tag{1}$$

For given $v_0 \in (b, \infty)$, integrating both sides of (1) from v_0 to v, where $v > v_0$, we have $2(\ln \ddot{p} - \ln \ddot{p}_0) \leq 3(\ln(-\dot{p}) - \ln(-\dot{p}_0))$, where $\ddot{p}_0 = \ddot{p}(v_0)$, $\dot{p}_0 = \dot{p}(v_0)$, then $\ddot{p}(\ddot{p}_0)^{-1} \leq (-\dot{p})^{3/2}(-\dot{p}_0)^{-3/2}$, or

$$\ddot{p}(-\dot{p})^{-3/2} \leq \ddot{p}_0(-\dot{p}_0)^{-3/2} \tag{2}$$

Integrating both sides of (2) from v_0 to v, we have

$$\begin{aligned} &2\left\{(-\dot{p})^{-1/2} - (-\dot{p}_0)^{-1/2}\right\} \leq \ddot{p}_0(-\dot{p}_0)^{-3/2}(v - v_0) \\ &\mu = (-\dot{p})^{1/2} \geq 2(-\dot{p}_0)^{3/2}(\ddot{p}_0(v - v_0) + 2(-\dot{p}_0))^{-1} \end{aligned} \tag{3}$$

From (3), we have

$$p \leq \frac{4(-\dot{p}_0)^3}{\ddot{p}_0} \frac{1}{\ddot{p}_0(v - v_0) + 2(-\dot{p}_0)} + p_0 - \frac{2(-\dot{p}_0)^2}{\ddot{p}_0} \tag{4}$$

where $p_0 = p(v_0)$. Similarly, when $v < v_0$, we get

$$\ddot{p}(\ddot{p}_0)^{-1} \geq (-\dot{p})^{3/2}(-\dot{p}_0)^{-3/2} \tag{5}$$

$$\mu = (-\dot{p})^{1/2} \geq 2(-\dot{p}_0)^{3/2}(\ddot{p}_0(v - v_0) + 2(-\dot{p}_0))^{-1} \tag{6}$$

$$p \geq \frac{4(-\dot{p}_0)^3}{\ddot{p}_0} \frac{1}{\ddot{p}_0(v - v_0) + 2(-\dot{p}_0)} + p_0 - \frac{2(-\dot{p}_0)^2}{\ddot{p}_0} \tag{7}$$

Let $\ddot{p}_0(\tilde{v} - v_0) + 2(-\dot{p}_0) = 0$, then $\tilde{v} = v_0 - 2\frac{(-\dot{p}_0)}{\ddot{p}_0}$. Since we suppose $\lim_{v \to b^+} p(v) = \infty$, in view of (7), we have $-\infty < \tilde{v} \leq b < v_0$. So b is a finite number.

Following Nishida [1], we have