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EXTINCTION PROPERTIES OF SOLUTIONS TO HYPERBOLIC EQUATIONS

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Abstract In this paper the extinction properties of solutions to both linear and nonlinear hyperbolic equations, which reflect the effect of damping boundary conditions and have not been explicitly stated in literature before, are introduced and extensively studied.

Key Words extinction properties of solutions; linear and nonlinear hyperbolic equations; damping boundary conditions.
Classifications 35B05; 35L20; 35L50.

1. Introduction

It is well known that some nonlinear parabolic equations have extinction properties, i.e., there exists a finite time $T > 0$ such that the solutions identically vanish as $t \geq T$ (see [1] and the references cited there). Such an extinction property reflects in some senses the effects of absorbing source of heat.

It was not known the literature whether the solutions to linear or nonlinear hyperbolic equations possess such a property. It is evident from the method of separate variables that the solutions to the wave equations with homogeneous Dirichlet or Neumann boundary conditions do not have such a property. Then one may look at the hyperbolic analog of that studied in [1]: namely

$$u_{tt} = u_{xx} + f(u) \tag{1.1}$$

with homogeneous Dirichlet boundary conditions at the boundary. Of course, it does not make sense to consider the problem with $f(u) = u^p$, $0 < p < 1$, because, as is well known, the solution to the initial boundary value problem of the wave equations does not hold the maximum principle which implies that the solution can be negative even for positive initial datum. On the other hand, if f is a locally Lipschitz continuous defined in R, then it is easy to prove by the contradiction arguments that the solution to (1.1) with homogeneous Dirichlet boundary conditions does not have the extinction property. Indeed, if there is a finite time $T > 0$ such that the solution identically vanishes as $t \geq T$ which implies $u(x, T) = u_t(x, T) = 0$, then solving the problem

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backward starting from $t = T$ yields that u identically vanishes even for $0 \le t \le T$ which contradicts with nonzero initial datum.

In this paper, however, we shall show that the solutions to the initial boundary value problem of both linear and nonlinear hyperbolic equations with certain damping boundary conditions do exist such an extinction property which, to our knowledge, has not been explicitly stated in the literature before. This property reflects the effects of damping boundary conditions.

More precisely, we consider in Section 2 the following problem

$$u_{tt} - u_{xx} = 0 \tag{1.2}$$

$$u(0,t) = 0, \ (u_t + \alpha u_x)(l,t) = 0 \tag{1.3}$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \tag{1.4}$$

with constant $\alpha > 0$.

The boundary condition at $x = l$ which is usually called damping boundary condition implies that there is friction on the boundary which is proportional to the velocity u_t. We then have

Theorem 1 *Suppose $\alpha = 1$. Then for any smooth initial data (u_0, u_1) the solution $u(x,t)$ to the above problem identically vanishes for $t \ge 2l$, i.e.,*

$$u \equiv 0 \quad \textit{for all } t \ge 2l, \ x \in [0,l]$$

The stability with respect to α is also proved in Section 2.

In Section 3 we are concerned with the extension to the linear symmetric hyperbolic system of first order with certain admissible boundary conditions. In Section 4 we consider the nonlinear wave equation

$$u_{tt} - (\sigma(u_x))_x = 0 \tag{1.5}$$

with the Dirichlet boundary condition at $x = 0$

$$u(0,t) = 0 \tag{1.6}$$

and the nonlinear damping boundary condition at $x = l$

$$(f(u_t) + \sigma(u_x))(l,t) = 0 \tag{1.7}$$

and the initial condition at $t = 0$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \tag{1.8}$$

where f and σ are smooth functions with $\sigma(0) = f(0) = 0, \ f'(0) > 0, \sigma'(s) > 0$.

It is known (see [2]–[4]) that if the initial data are appropriately small, then problem (1.5)–(1.8) admits a unique global classical solution which decays to zero exponentially as $t \to +\infty$.