J. Partial Differential Equations
Vol. 4, *No.* 2 (1991), 74–96.

THE ONE-DIMENSIONAL QUASILINEAR VERIGIN PROBLEM

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(Received May 5, 1989; revised Oct. 3, 1989)

Abstract In this paper, we consider the one-dimensional quasilinear Verigin problem and the relative diffraction problem. The existence, uniqueness, and the smoothness of the global solution are discussed.

Key words. verigin displacement; quasilinear; free boundary; interface; diffraction; existence; uniqueness; smoothness.

Classification 35K65.

0. Introduction

In the production, we often meet the piston-like displacement of one fluid to another in porous media. For example, there is a displacement of oil by water in recovering petroleum and the displacement of water by slurry in construction. This kind of model can be described as a free boundary problem in mathematics. When the porous media is compressible, Verigin put forth the mathematical model of this kind of problem in 1957 as Muskat Problem[15]. We call it Verigin Problem[1].

In one-dimensional case, there are many results about the linear problems. First, Rubinstein proved the existence of the local solution by making use of the potential method and fixed point theorem in 1957[2]. Then, Kamynin gave the proof of the existence of the global solution under some assumptions which guaranteed that the free boundary was monotonous in 1963[3,4]. Unfortunately, his estimate of the first derivative had a mistake. In 1969, Fulk and Guenther simplified the proof of Rubinstein, and gave a new proof of the existence and uniqueness of the local solution[5]. In 1977–1980, Evans and Friedman etc. published a series of papers in which the existence, uniqueness, smoothness and asymptotic behavior of the global solution were discussed and proved without the supposal of $\mu_1 > \mu_2$[6–9]. Until that time, the results of the research on the linear problems were quite ripe. In 1980, Meirmanov considered a porous medium equation in a special form as follows

$$\begin{cases} m\rho_{it} + (\rho_i v_i)_x = 0 \\ v_i = \dfrac{k}{\mu_i} p_{ix} \\ \rho_i = \lambda_i(p_i + a_i), \quad i = 1,2 \\ \text{free boundary condition} \end{cases} \tag{0.1}$$

and proved the existence and uniqueness of the solution with some strong conditions[10].

In this paper, we have discussed the one-dimensional quasilinear Verigin Problem, whose form is much more general then (0.1). We have proved the existence, uniqueness and smoothness of the global solution. Here our method is much different from Meirmanov's.

In order to consider Verigin Problem, we need to discuss carefully the diffraction problems which have a close relation with Verigin Problem. There are already many works about the diffraction problems. The representative works are made by Olenik, Ladyzenskaja and Kamynin etc., [12–14]. In this paper, we obtain some more sharp results about the quasilinear parabolic diffraction problems. Here, the existence, uniqueness and smoothness of the global solution are proved. Especially, the behavior of the solution near the interface is discussed at length. Also, we get the continuity of the solution of the diffraction problem about perturbation of its interface.

This paper is divided into two parts. Part 1 is about the quasilinear parabolic diffraction problem. In Part 2, the Verigin Problem is considered.

Part 1
One-dimensional Quasilinear Parabolic Diffraction Problem

In this part, we consider the quasilinear parabolic diffraction problem as follows

$$\begin{cases} \Phi(p)_t - (K(s_f(x,t))p_x)_x = 0 \quad \text{in} \quad Q_\tau \\ p|_{x=0} = \bar{p}_1(t), \quad p|_{x=b} = \bar{p}_2(t) \\ p|_{t=0} = \bar{p}_0(x) \end{cases} \tag{1.1}$$

where $Q_\tau = (0,b) \times (0,\tau)$, it is divided into two parts of Q_1 and Q_2 by an interface $\Gamma : x = f(t)$; $\Phi(t)$ is a nonlinear function see Figure 1. $K(s) = \left(\dfrac{k}{\mu_1} - \dfrac{k}{\mu_2}\right)s + \dfrac{k}{\mu_2}$;