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BLOW-UP OF SOLUTIONS OF A CLASS OF NONLINEAR PARABOLIC EQUATIONS

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Abstract In this paper we give a detailed discussion about the effect of quantitative relation between p and m on the properties of the solutions to nonlinear parabolic equation $u_t - (u^m u_x)_x = u^p$.

Key Words Global solution; local solution; blow up; blow-up set.
Classifications 35A05; 35B; 35K.

0. Introduction

In this paper we consider the initial-boundary value problem

$$\left\{\begin{array}{ll} u_t - (u^m u_x)_x = u^p & -R < x < R,\ t > 0 \\ u(x,0) = \varphi(x) & -R < x < R \\ u(\pm R, t) = 1 & t > 0 \end{array}\right. \tag{0.1}$$

where $p > 0$ is a positive constant, $m \geq 1$ is an integer. We set

$$Q_T = (-R, R) \times (0, T)$$

We assume throughout this paper that

$$\begin{array}{ll} \varphi(x) \in C^{2+\alpha}([-R, R]), & \text{for some } \alpha > 0 \\ \varphi(x) \geq 1,\ \varphi(\pm R) = 1 & \end{array} \tag{0.2}$$

$$-\frac{d}{dx}\Big[\varphi^m \frac{d\varphi}{dx}\Big]\Big|_{x=\pm R} = 1 \tag{0.3}$$

The significance of this problem lies in the fact that if $u > 1$, then the effect of diffusion coefficient u^m and reaction term u^p on the blow-up properties of the solutions of (0.1) is just the opposite. Roughly speaking, if m is sufficiently large, then the diffusion term u^m plays a main role and the solution does not blow up in a finite time, i.e., there exists a global solution of (0.1). On the contrary, if p is sufficiently large, then the reaction term u^p plays a main role and the solution blows up in a finite time, i.e., there exists only a local solution of (0.1). Naturally, we hope to give a quantitative relation between p and m to decide whether the solutions of (0.1) will blow up or not.

In addition, we will study the blow-up set and give a rough estimate of blow-up time. Our main results are:

(i) If $p < m+1$ then there exists a unique global solution u of (0.1).

(ii) If $p > m+1$ then there exists a unique solution u of (0.1) for $0 < t < T$, where T is a finite time, and u blows up as $t \uparrow T$ if $\varphi(x)$ is sufficiently large. And we prove, if φ has just one point of maximum, then the blow-up set consists of a single point.

(iii) If $p = m+1$, $R \le \pi/2\sqrt{m+1}$ then there exists a unique global solution u of (0.1). If $R > \pi/2\sqrt{m+1}$ then the solution u of (0.1) blows up in a finite time and the blow-up set is $S = \left\{-\frac{\pi}{2\sqrt{m+1}} \le x \le \frac{\pi}{2\sqrt{m+1}}\right\}$.

1. Existence Theorem

Theorem 1.1 *Assume that* (0.2) *and* (0.3) *hold. If* $p < m+1$ *then there exists a unique global solution of* (0.1). *If* $p \ge m+1$ *then there exists a unique local solution of* (0.1).

Proof Let $v = u^{m+1}$ then we have

$$\begin{cases} v^{-m/(m+1)}v_t - v_{xx} = (m+1)v^{p/(m+1)} & -R < x < R,\ t > 0 \\ v(x,0) = \varphi^{(m+1)}(x) & -R < x < R \\ v(\pm R, t) = 1 & t > 0 \end{cases} \tag{1.1}$$

Thus, it suffices to prove the theorem for (1.1). To do this, we shall apply Schauder fixed point theorem.

First, we prove the existence of global solution. Let $T > 0$ be any positive real number. We introduce the Banach space $B = C^{\beta,\frac{\beta}{2}}(Q_T)$, where $\beta > 0$ to be determined, and define a mapping S as follows. For each $w \in B, w \ge 1$ let $v = Sw$ be the solution of the following problem

$$\begin{cases} w^{-m/(m+1)}v_t - v_{xx} = (m+1)v^{p/(m+1)} & -R < x < R,\ 0 < t < T \\ v(x,0) = \varphi^{m+1}(x) & -R < x < R \\ v(\pm R, t) = 1 & 0 < t < T \end{cases} \tag{1.2}$$

First, we prove that there exists a unique solution $v(x,t) \in C^{2+\beta,1+\frac{\beta}{2}}(\bar{Q}_T)$ of (1.2). It suffices to construct a pair of supersolution and subsolution of (1.2). Obviously, $\underline{v} \equiv 1$ is a subsolution of (1.2). Next, we take $b > R$ and consider the problem

$$\begin{cases} -\psi'' = \lambda\psi & -b < x < b \\ \psi(\pm b) = 0 \end{cases} \tag{1.3}$$

Its first eigenvalue is $\lambda = \pi^2/4b^2$ and first eigenfunction is $\psi(x) = \cos(x/2b)$. It is easily seen that $c_0 = \min\limits_{[-R,R]} \psi(x) = \cos(R/2b) > 0$. Setting $M = \max\limits_{[-R,R]} \varphi(x) \ge 1, q = p/(m+1) < 1$ and taking $d = \max\{[(m+1)/\lambda]^{1/(1-q)}, M^{m+1}\}$, then we can check that $\bar{v} = (d/c_0)\psi(x)$ is a supersolution of (1.2). Hence, there exists a unique solution