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ON NONLINEAR EIGEN-PROBLEMS OF QUASI-LINEAR ELLIPTIC OPERATORS*

Ma Li
(Institute of Math., Peking University, 100871, China)
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Abstract In this paper, we study the following Eigen-problem

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\Big(a_{ij}(x,u)\dfrac{\partial u}{\partial x_j}\Big)+\dfrac{1}{2}a_{iju}(x,u)\dfrac{\partial u}{\partial x_i}\dfrac{\partial u}{\partial x_j}+h(x)u=\mu u^{\frac{n+2}{n-2}} & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \\ u>0 & \text{in } \Omega\subset R^n \end{cases} \tag{0.1}$$

under some assumptions. First we minimize

$$I(u)=\frac{1}{2}\int_\Omega a_{ij}(x,u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+h(x)u^2$$

over

$$E_\alpha=\Big\{u\in H_0^1(\Omega);\int_\Omega u^\alpha=1\Big\}\quad\Big(2<\alpha<N=\frac{2n}{n-2}\Big)$$

to give a H_0^1-solution U_α of the perturbation problems of (0.1). Since I is not differentiable in $H_0^1(\Omega)$, the key point is the estimate of U_α. Then, we derive local uniform bounds of (U_α) and give a 'bad' solution of (0.1). Last, we remove the singular points of the 'bad' solution to obtain a solution of (0.1), our result is a extension of that of Brezis & Nirenberg.

Key Words Non-differentiable; critical; regularity.
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1. Introduction

Suppose Ω is a bounded domain of $R^n (n\geq 3)$ with smooth boundary. We will study the following nonlinear eigen-problems: Find a pair $(\lambda,u)\in R\times C^2(\Omega)$ such that

$$-\frac{\partial}{\partial x_i}\Big(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\Big)+\frac{1}{2}a_{iju}(x,u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+h(x)u=\lambda u^\alpha\quad \text{in } \Omega \tag{1.1}$$

$$u>0\quad \text{in } \Omega \tag{1.2}$$

*This is a part of the author's Master thesis in Nankai University.

$$u = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

where $h(x)$ is a continuous function, $a_{ij}(x,u)$ is a continuously differentiable function on $\Omega \times R$ with following conditions: there exist positive constants $\Lambda_1 \le 1, \Lambda_* < 1$ such that

$$\begin{cases} \text{(a) } \Lambda_1 \le a_{ij}(x,u)p_ip_j \le \Lambda_1^{-1} & \text{for all } (x,u,p) \in \Omega \times R^+ \times S^{n-1} \\ \text{(b)} -\dfrac{u}{2}a_{iju}(x,u)p_ip_j \le \Lambda_* a_{ij}(x,u)p_ip_j & \text{for all } (x,u,p) \in \Omega \times R^+ \times S^{n-1} \end{cases} \tag{1.4}$$

$\alpha \in \left(1, \dfrac{n+2}{n-2}\right]$. We have used Einstain sum convention for i, j with the range from 1 to n and only use this in the follows. Formally, (1.1) is the Euler-Lagrange's equation of the functional

$$I(u) = \frac{1}{2}\int_\Omega a_{ij}(x,u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + h(x)u^2$$

over the set

$$\mathcal{M}_\alpha = \left\{u \in H^1(\Omega);\ \int_\Omega u^{\alpha+1} = 1\right\}$$

But $I(\cdot)$ is not differentiable on H^1! The functional $I(\cdot)$ is only differentiable on $H^1 \cap L^\infty$, which is a good space in this variational problems. When $\alpha = \dfrac{n+2}{n-2} := \alpha_0$, another difficulty in solving (1.1–1.3) is that the Sobolev's imbedding $H^1(\Omega) \hookrightarrow L^{\alpha_0+1}(\Omega)$ is not compact.

Since we are only seeking positive solutions, without loss of generality, we assume that $a_{ij}(x,u)$ are even in u, i.e. $a_{ij}(x,u) = a_{ij}(x,-u)$.

Our main theorems are

Theorem 1 *For $\alpha \in \left(1, \dfrac{n+2}{n-2}\right)$, if the assumption (1.4) holds, then there exists a pair $(\lambda_\alpha, u_\alpha) \in R \times C^2(\Omega)$ satisfying (1.1–1.3). Moreover, u_α minimizes $I(\cdot)$ on $\mathcal{M}_\alpha$.*

Theorem 2 *For $\alpha_0 = \dfrac{n+2}{n-2}$. If the condition (1.4) holds with*

$$u a_{iju}(x,u) \to 0 \quad (u \to +\infty) \text{ uniformly in } \quad x \in \bar\Omega \tag{*}$$

and

$$\inf_{u\in\mathcal{M}_{\alpha_0}} I(u) < \frac{1}{2} S \sqrt[n]{\inf_{x\in\Omega} \det(a_{ij}(x))} \tag{**}$$

where

$$a_{ij}(x) = \lim_{u\to+\infty} a_{ij}(x,u)$$

is a continuous function and S is the best Sobolev constant in R^n. Then there exists a pair $(\lambda, u) \in R \times C^2(\Omega)$ satisfying (1.1–1.3).

Our plan of proofs is as follows. To prove Theorem 1, we first get a minimizer of $I(\cdot)$ on $\mathcal{M}_\alpha$. Then we construct a special test space for this minimizer to differentiate