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UNIQUENESS OF SOLUTION OF THE INITIAL VALUE PROBLEM FOR $u_t = \Delta u^m - u^p$

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Abstract The uniqueness of solution of the Cauchy problem

$$u_t = \Delta u^m - u^p, \qquad S_T = R^n \times (0,T)$$

$$u(x,0) = \phi(x), \qquad x \in R^n$$

is obtained. Where $n \geq 1, m, p > 0, \phi(x) \in L^\infty(R^n), \phi(x) \geq 0$.

Key Words Uniqueness of solution; initial value problem.
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1. Introduction

In this paper we consider the Cauchy problem

$$u_t = \Delta u^m - u^p, \qquad S_T = R^n \times (0,T) \tag{1.1}$$

$$u(x,0) = \phi(x), \qquad x \in R^n \tag{1.2}$$

where $n \geq 1, m, p > 0, \phi(x) \in L^\infty(R^n), \phi(x) \geq 0$.

Equation (1.1) arises in many applications. We will not recall them here, since they can be found in most of papers, for example [1]. For the case of regular diffusion ($m = 1$) and slow diffusion ($m > 1$), it was shown in [2] that the problem (1.1), (1.2) has a unique continuous solution, when $n = 1$. The object of this paper is to extend these results to the case when $m > 0, p > 0, n \geq 1$.

Let $B_R = \{x \in R^n : |x| < R\}$, $\partial B_R = \{x \in R^n : |x| = R\}$.

Definition *We say that a function* $u : S_T \to R$ *is a solution of* (1.1), (1.2), *if* $u \in L^\infty(S_T)$, $u \geq 0$ *and for almost all* $R > 0$, $t \in (0,T)$ *it satisfies the identity*

$$\int_{B_R} u(x,t)\xi(x,t) - \int_{B_R} \phi\xi(x,0) = \int_0^t \int_{B_R} (u\xi_t + u^m \Delta\xi - u^p\xi) - \int_0^t \int_{\partial B_R} u^m(\nabla\xi, \nu)$$

where $\xi \in C^2(\bar{S}_T)$, $\xi = 0$ *on* $\partial B_R \times (0,T)$, ν *denotes the outward pointing normal on* ∂B_R.

Our results are as follows

Theorem 1. *Let* $\phi \in L^\infty(R^n), \phi \geq 0$. *Assume that one of the following conditions holds*

(1) $0 < p < 1$, $m > \left(1 - \frac{2}{n}\right)^+ p$

(2) $p \geq 1$, $m > \left(1 - \frac{2}{n}\right)^+$

where $(s)^+ = \max\{s, 0\}$. *Then* (1.1), (1.2) *has a unique solution.*

Theorem 2. *Let* u_1, u_2 *be the solutions of* (1.1) *with nonnegative initial value* $\phi_1, \phi_2 \in L^\infty(R^n)$. *And assume that the conditions of Theorem* 1 *hold. Then* $\phi_1 \leq \phi_2$ *on* R^n *implies* $u_1 \leq u_2$ *on* S_T.

2. The Proof of Theorem

The proof of Theorem 1 We consider the approximate problem

$$u_t = \Delta u^m - u^p + \varepsilon^p \qquad \text{in} \quad B_{R(\varepsilon)} \times (0,T) \tag{2.1}$$

$$u(x,t) = \varepsilon \qquad \text{on} \quad \partial B_{R(\varepsilon)} \times (0,T) \tag{2.2}$$

$$u(x,0) = \phi_\varepsilon(x) \qquad \text{in} \quad B_{R(\varepsilon)} \tag{2.3}$$

where

$$0 < \varepsilon < 1,\ R(\varepsilon) = \varepsilon^{-\frac{p}{n}+\varepsilon_0},\ \frac{p-1}{n} < \varepsilon_0 < \frac{p}{n} - \frac{1}{2}(1-m)^+ \text{ when } p \geq 1$$

$$0 < \varepsilon_0 < \frac{p}{n} - \frac{1}{2}(p-m)^+ \quad \text{when} \quad 0 < p < 1$$

$\phi_\varepsilon \in C^\infty(R^n)$ has the properties

(1) $\phi_\varepsilon \geq \varepsilon$, $\int_{B_{R(\varepsilon)}} |\phi_\varepsilon - \phi| \to 0$ as $\varepsilon \to 0^+$

(2) $\phi_\varepsilon = \varepsilon$ near $|x| = R(\varepsilon)$

Remark ϕ_ε can be chosen in the following fashion. Let $\phi_\varepsilon = (\phi + \varepsilon) * J_{h(\varepsilon)}$ where $J_h \in C^\infty(R^n)$ is a mollifier with the properties $\mathrm{supp} J_h \subset \{x : |x| < h\}$, $\int_{R^n} J_h = 1$. Since $\varepsilon_0 > \frac{(p-1)^+}{n}$, we can choose $h(\varepsilon)$ such that

$$\phi_\varepsilon \geq \varepsilon, \quad \left\{\int_{R(\varepsilon)} |\phi * J_{h(\varepsilon)} - \phi| + \int_{R(\varepsilon)} \varepsilon * J_{h(\varepsilon)}\right\} \to 0 \quad \text{as} \quad \varepsilon \to 0^+$$

It is well known that (2.1)–(2.3) has a unique classical solution u_ε and $\varepsilon \leq u_\varepsilon \leq M$. Hence the uniform upper bound implies, by [3] and [4], that $\{u_\varepsilon\}$ is equicontinuous on