J. Partial Differential Equations
Vol. 4, No. 4 (1991), 1–8

AN EIGENVALUE PROBLEM ON NEGATIVELY CURVED MANIFOLDS

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(Received April 5, 1990)

Abstract Consider the eigenvalue problem:

$$\Delta_g u - \lambda K u = 0 \quad \text{in } D$$

where D is the unit disc of the complex plane, g is a complete metric conformal to the Poincaré metric on D, and K is the Gaussian curvature. It is shown that if $\lambda > \frac{1}{2}$ ($\lambda > \frac{1}{4}$ in the case of $K \leq 0$), then the above problem has no positive solutions.
Key Words Eigenvalue problem; complete manifolds; negative curvature.
Classifications 58G25; 58E12.

1. Introduction

Let $dg^2 = \mu^2|dz|^2$ be a complete metric on the unit disc D of the complex plane, and let $K = -\Delta \log \mu/\mu^2$ be the corresponding intrinsic Gauss-curvature. We are interested in the following eigenvalue problem:

$$\Delta_g u - \lambda K u = 0 \quad \text{in } D \tag{1.1}$$

The motivation of the study of (1.1) comes from the classification of complete, oriented, immersed surfaces in R^3 which are stable for an elliptic parametric integral Φ. (See e.g. [4]). Indeed, let M be a complete, oriented, immersed surface in R^3 which is stable for an elliptic parametric integral Φ with constant coefficients, then, as in [3], one may argue that $\tilde{M}$, the universal cover of M, is also Φ-stable. On the other hand, $\tilde{M} \hookrightarrow R^3$ is Φ-stable if and only if there is a positive entire solution, u, of

$$\operatorname{div}_{\tilde{M}}(\Sigma \circ \nabla_{\tilde{M}} u) + (\Sigma \circ A^2)u = 0 \quad \text{in } \tilde{M} \tag{1.2}$$

Here A is the second fundamental form of M in R^3, and $\Sigma : TM \to TM$ is a positive definite linear operator (depending only on v_M, the unit normal vector on M) varying smoothly with v_M.

* This subject partially supported by the NSF Grant.

In (1.2), A and Σ should be thought of as defined on $\tilde{M}$ by Deck transformations. In the case of the area integral, (1.2) becomes

$$\Delta_{\tilde{M}}u + |A|^2 u = 0 \quad \text{in } \tilde{M} \tag{1.3}$$

If u is a positive solution of (1.3), then $\Delta_{\tilde{M}}u \leq 0$ in $\tilde{M}$ and equality holds identically if and only if $|A| \equiv 0$, i.e. M is a plane. In the case that $\Delta_{\tilde{M}}\overline{u} \leq$ and $\not\equiv 0$, the conformal type of $\tilde{M}$ must be D since there is no positive superharmonic function in the complex plane. Thus the problem (1.3) is equivalent to (1.1) with $\lambda = 2$.

In general the existence of a positive solution of (1.2) will imply the existence of a positive solution of (1.1) for some $\lambda > 0$ (See discussion below).

It was shown by Do Carmo and Peng [1] that there is no positive solution to (1.1) for $\lambda \geq 2$. Independently, Fischer-Colbrie and Schoen [3] proved a slightly more general theorem which implies, in particular, that there is a number a_0; $0 \leq a_0 \leq 1$, such that (1.1) has no positive solution if $\lambda > a_0$ and that (1.1) does possess a positive solution if $\lambda \leq a_0$.

If one considers the Poincaré metric on D with $K \equiv -1$ and $dg^2 = \dfrac{4}{(1-r^2)^2}|dz|^2$, $r^2 = |z|^2$, then it is easy to check that $a_0 = \dfrac{1}{4}$. (See also [5] for the related discussion).

In this paper, we shall prove the following

Theorem A *Let $dg^2 = \mu^2|dz|^2$ be a complete metric on D. Then for $\lambda > 1/2$ there is no positive solution to* (1.1). *If, in addition, the Gaussian curvature $K \leq 0$, then there will be no positive solution to* (1.1) *for $\lambda > \dfrac{1}{4}$.*

Theorem A clearly exhibits another extreme property of the Poincaré-metric. There is also a higher dimensional version of Theorem A (See Section 3 below).

As a consequence of Theorem A, we have

Corollary *Let $M \hookrightarrow R^3$ be a complete, oriented, immersed surface which is stable for an elliptic parametric integral Φ with constant coefficients. Suppose, in addition*

$$\|\Phi - 1\|_{C^{2,\alpha}} \leq \varepsilon_0 \tag{1.4}$$

for some positive constant ε_0 (which is not too small!). Then M is a plane.

We do not know if (1.4) is necessary. Some open problems related to this are discussed at the end of the paper.

The results in this paper were proven in early 1984. The delay in its publication is mainly due to condition (1.4) above which the author believes is not necessary. It remains, however, an open problem.