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DIFFERENCE SCHEMES OF FULLY NONLINEAR PARABOLIC SYSTEMS OF SECOND ORDER

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Abstract The general difference schemes for the first boundary problem of the fully nonlinear parabolic systems of second order

$$f(x,t,u,u_x,u_{xx},u_t)=0$$

are considered in the rectangular domain $Q_T=\{0\le x\le l,\ 0\le t\le T\}$, where $u(x,t)$ and $f(x,t,u,p,r,q)$ are two m-dimensional vector functions with $m\ge 1$ for $(x,t)\in Q_T$ and $u,p,r,q\in R^m$. The existence and the estimates of solutions for the finite difference system are established by the fixed point technique. The absolute and relative stability and convergence of difference schemes are justified by means of a series of a priori estimates. In the present study, the existence of unique smooth solution of the original problem is assumed. The similar results for nonlinear and quasilinear parabolic systems are also obtained.

Key words Fully nonlinear parabolic system, difference scheme, stability, convergence.

Classication 65N10.

1. Introduction

1. In [1–5], for the boundary problem of the quasilinear parabolic systems of higher order

$$(-1)^M u_t + A(x,t,u,u_x,\cdots u_{x^{M-1}})u_{x^{2M}} = f(x,t,u,u_x,\cdots,u_{x^{2M-1}}) \tag{1}$$

some general finite difference schemes are studied, where $u=(u_1,\cdots,u_m)(m\ge 1)$ and $f(x,t,u,u_x,\cdots,u_{x^{2M-1}})$ are m-dimensional vector functions, $A(x,t,u,u_x,\cdots,u_{x^{M-1}})$ is a $m\times m$ positively definite matrix and $M\ge 1$ is an integer. The absolute and relative convergence of the solutions of the difference schemes to the generalized vector solution $u(x,t)\in W_2^{(2M,1)}(Q_T)$ of the original problem have been established in the sense of weak convergence for the functional space $W_2^{(2M,1)}(Q_T)$.

In the present work, we are going to study the difference schemes for the fully nonlinear parabolic systems of second order

$$f(x,t,u,u_x,u_{xx},u_t)=0, \tag{2}$$

where $u = (u_1, \cdots, u_m)(m \geq 1)$ and $f(x,t,u,p,r,q)$ are the m-dimensional vector functions and $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ are the corresponding vector derivatives.

The fully nonlinear systems (2) of second order is said to be uniformly parabolic if for the $m \times m$ Jacobi derivative matrices $f_q(x,t,u,p,r,q)$ and $f_r(x,t,u,p,r,q)$, there is a positive constant $a > 0$, such that

$$(\xi, -f_q^{-1} f_r \xi) \geq a|\xi|^2 \tag{3}$$

for any $\xi \in R^m$, where $(x,t) \in Q_T$ and $u,p,r,q \in R^m$. Let us consider in the rectangular domain $Q_T = \{0 \leq x \leq l, 0 \leq t \leq T\}$ with $l > 0$ and $T > 0$, the problem of system (2) with the boundary conditions

$$u(0,t) = \psi_0(t), \quad u(l,t) = \psi_1(t) \tag{4}$$

and the initial condition

$$u(x,0) = \phi(x) \tag{5}$$

where $\psi_0(t)$, $\psi_1(t)$ and $\phi(x)$ are given m-dimensional vector functions of variables $t \in [0,T]$ and $x \in [0,l]$ respectively.

For the boundary problems (4) and (5) of the nonlinear and quasilinear parabolic systems of second order

$$u_t = f(x,t,u,u_x,u_{xx}) \tag{6}$$

and

$$u_t = A(x,t,u,u_x)u_{xx} + F(x,t,u,u_x) \tag{7}$$

some general finite difference schemes are also studied by the similar method and analogous results are obtained.

For the sake of brevity, we adopt the similar notations and abbreviations are used in [1,5] and some lemmas and technique of treatment in [1,5] are repeately used in the present work.

2. Let us divide the rectangular domain Q_T into small grids by the parallel lines $x = x_j$ $(j = 0,1,\cdots,J)$ and $t = t^n$ $(n = 0,1,\cdots,N)$ with $x_j = jh$ and $t^n = n\tau$, where $Jh = l$ and $N\tau = T$, J and N are integers, and h and τ are the steplengths of the grids. Denote $v_\Delta = \{v_j^n | j = 0,1,\cdots,J;\ n = 0,1,\cdots,N\}$ the m-dimensional discrete vector function defined on the grid points $\{(x_j,t^n)|\ j = 0,1,\cdots J; n = 0,1,\cdots,N\}$.

Let us now construct the general finite difference schemes of the above mentioned system (2) as follows:

$$f\left(x_j, t^{n+\alpha}, \bar{\delta}_0 v_j^{n+\alpha}, \bar{\delta}^1 v_j^{n+\alpha}, \frac{\Delta_+\Delta_- v_j^{n+\alpha}}{h^2}, \frac{v_j^{n+1} - v_j^n}{\tau}\right) = 0$$

$$(j = 1,2,\cdots,J-1; n = 0,1,\cdots,N-1) \tag{2_h}$$