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DIRICHLET PROBLEM FOR DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS SUGGESTED BY THE ANISOTROPIC PERMEATION*

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Abstract By Browder's pseudo-monotone operator theory and the techniques belonging to J. Leray and J. Lions, the existence theorem of the generalized solution of the Dirichlet problem for a strongly degenerate quasilinear elliptic equation has been proved in the anisotropic Sobolev space.
Key Words Degenerate equation; imbedding theorem; anisotropic permeation; pseudo-monotone operator.
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1. Introduction

For the studies on permeating theory so far many papers have been devoted. But only few articles are on the degenerate permeating problem, especially on the anisotropic permeating problem. Paper [1] studied a simple case for the degenerate permeating problem and some authors generalized its result to the same kind of problems, in which the equations are not strongly degenerate and the permeation is not anisotropic. For a class of anisotropic permeating problems, the mathematical model is the Dirichlet problem for a strongly degenerate quasilinear elliptic equation in the anisotropic Sobolev space. The character of this kind of permeating problems is that the pressure function $u(x)$ of concerned fluid is not homogeneous in directions. The mathematical sense for this case is that the gradient Du has different exponents of integrability for its different components. In this paper we formulate the mathematical model for this kind of permeating problems and prove the existence of its generalized solution. The method we used to prove our main theorem is of pseudo-monotone (see [2]) and the techniques belong to J. Leray and J. Lions (see Chapter 5 in [3]).

Suppose that the velocity function of concerned non-compressible fluid is as follows

$$\vec{V} = -(a_1(x,u,Du), a_2(x,u,Du), \cdots, a_n(x,u,Du)),$$

where $a_i, i = 1, 2, \cdots, n$, are known functions of their independent variables; u is the pressure function to be solved.

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The mathematical model for the anisotropic permeating problem is as follows

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(a_i(x,u,Du))+a_0(x,u,Du)=f(x),\quad x\in\Omega \tag{1}$$

$$u=0,\quad x\in\partial\Omega \tag{2}$$

where $\Omega\subset R^n$ $(n\geq 3)$ is a bounded domain in which Sobolev imbedding theorem (Lemma 1) is valid.

The degenerality of equation (1) is that the usual monotonic condition restricting $a_i(x,u,Du)$, $i=1,2,\cdots,n$, does no longer hold. At the same time $D_iu, i=1,2,\cdots,n$, possess varying integrability with their indices i.

2. Main Theorem

Let $\Omega\subset R^n$ $(n\geq 3)$ be a bounded domain. Define a norm in $C_0^1(\Omega)$ as below

$$\|u\|_{\overset{\circ}{W}{}^1_m(\Omega)}=\sum_{i=1}^{n}\|D_iu\|_{m_i} \tag{3}$$

we call the closure of $C_0^1(\Omega)$ with the norm in (3) a anisotropic Sobolev space written by $\overset{\circ}{W}{}^1_m(\Omega)$ (see [4]) where $m_i\geq 2$, $m=(m_1,m_2,\cdots,m_n)$.

Definition 1 $u\in\overset{\circ}{W}{}^1_m(\Omega)$ *is a generalized solution of problem* (1) *with* (2) *if it satisfies for any* $v\in\overset{\circ}{W}{}^1_m(\Omega)$ *the following identity*

$$\int_\Omega\Big(\sum_{i=1}^{n}a_i(x,u,Du)D_iv+a_0(x,u,Du)v\Big)dx=\int_\Omega f(x)vdx \tag{4}$$

Suppose that the following conditions hold:

(A_1) $a_i(x,z,p)$, $i=0,1,\cdots,n$, are continuous in $(z,p)\in R\times R^n$ for almost every $x\in\Omega$ and Lebesgue measurable in x for every $(z,p)\in R\times R^n$.

(A_2) The following structure conditions hold:

$$|a_i(x,z,p)|\leq c_1(|p_i|^{m_i-1}+|z|^{q/m_i'}+k_1(x)),\quad i=1,2,\cdots,n$$

$$|a_0(x,z,p)|\leq c_2\Big(\sum_{i=1}^{n}|p_i|^{m_i/q'}+|z|^{q-1}+k_2(x)\Big)$$

where $k_1(x)$, $k_2(x)>0$; $k_1(x)\in L_{m^*}(\Omega)$, $m^*=\max\{m_1,m_2,\cdots,m_n\}$; $k_2(x)\in L_{q'}(\Omega)$, q' is the conjugate number of q with $q>0$ and $\frac{n}{q}>\sum_{i=1}^{n}\frac{1}{m_i}-1$.

(A_3) Instead of O. A. Ladyzhenskaya's monotonic condition (see [5]) we give the strongly degenerate condition as follows for every $\xi=(\xi_1,\xi_2,\cdots,\xi_n)\in R^n$ it holds

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial p_j}a_i(x,z,p)\xi_i\xi_j\geq\nu(x,z)g(|p|)\sum_{i=1}^{n}|p_i|^{m_i-2}|\xi|^2$$