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ON THE CAUCHY PROBLEM FOR THE EQUATION OF FINITE-DEPTH FLUIDS*

Zhou Yulin, Guo Boling
and Tan Shaobin
(Institute of Applied Physics and Computational Mathematics, Beijing, 100088)
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Abstract Some properties of the singular integral operator $G(\cdot)$ and the solvability of Cauchy problem for the singular integral-differential equations (1.1) and (1.2) of finite-depth fluids are studied.

Key Words Singular integral differential equation; Joseph equation; Cauchy problem; *a priori* estimates; Solvability.

Classification 35Q.

1. Introduction

In this paper we consider the Cauchy problem for the equation of finite-depth fluids [1]

$$U_t + 2UU_x - G(U_{xx}) = 0, \quad t \geq 0, x \in R \tag{1.1}$$

which was proposed by Joseph (1977) and later derived by Kubota and Dobbs (1978), where $U_t = \partial U/\partial t, U_x = \partial U/\partial x$ and etc. We also consider the generalized equation of finite-depth fluids with diffusion term

$$U_t = \alpha U_{xx} + \beta G(U_{xx}) + \varphi_x(U) \tag{1.2}$$

where $\varphi(\cdot)$ is assumed to be a mildly smooth function on R, such that

$$|\varphi^{(j)}(u)| \leq C(1 + |u|^{3-j}) \text{ for } j = 0, 1, \ u \in R \tag{A}$$

$$G(u) = P. \int_{-\infty}^{\infty} \frac{1}{2\delta}\Big(\coth \frac{\pi(x-y)}{2\delta} - \operatorname{sgn}(x-y)\Big) U(y) dy$$

is a singular integral operator; P. denotes the Cauchy principal value; α, β, δ are constants with $\alpha \geq 0, \delta > 0$. Equation (1.1) appears in the studying of oceanics and atmospheric science, which describes the evolution of long internal waves with small

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amplitude in a stably stratified fluid, propagating in one direction. The constant δ expresses the degree of depth. As the depth of fluid approaches to zero, equation (1.1) approximates to the well-known KdV equation [2]

$$U_t + 2UU_x + U_{xxx} = 0 \tag{1.3}$$

On the other hand, as δ tends to infinite, Equation (1.1) then reduces to the Benjamin-Ono equation

$$U_t + 2UU_x + H(U_{xx}) = 0 \tag{1.4}$$

where

$$H(U) = P. \int_{-\infty}^{\infty} \frac{U(y)}{\pi(y-x)} dy$$

is the Hilbert transform.

The KdV equation (1.3) and BO equation (1.4) have been studied extensively in their relation to the existence of a unique smooth solutions, the asymptotic behavior and soliton solution problems. According to the equation of finite-depth fluids (1.1) there are a few works [1–6] which concerned with the integrability and the solitary solutions, but the solvability of Cauchy problem has not been found to be discussed. In the present paper we shall concentrate on the Cauchy problem for the equation of finite-depth fluids (1.1), and the Cauchy problem for the generalized equation (1.2). Roughly speaking, we shall first study some properties of the singular integral operator $G(\cdot)$, then with the help of these properties we shall demonstrate the following results: in H^s with $s \geq 2$, the Cauchy problem for Equation (1.2) with $\alpha > 0$ is global well-posed and Equation (1.2) with $\alpha = 0$ is locally well-posed in a classical sense. For H^1, the Cauchy problem for the equation of finite-depth fluids (1.1) has at least one global solution in a weak class $L^\infty(0,T;H^1)$ for every initial data $U_0(x)$ given in H^1.

2. Preliminaries

We introduce the following notation: By $L^p(R), H^s(R)$ and $W_p^s(R)$ we denote the usual Sobolev spaces, the relative norms denoted respectively by $\|\cdot\|_p, \|\cdot\|_{H^s}$ and $\|\cdot\|_{W_p^s}$, where $p \geq 1$ is a real number, $s \geq 0$ is an integer number. By $W_p^{s,[\frac{s}{2}]}(Q_T)$ we denote the space of function $f(x,t)$ which has derivatives $D_t^r D_x^k f(x,t) \in L^p(Q_T)$ with $2r + k \leq s$, where $Q_T = R \times [0,T]$, T is an arbitrary positive number. By $W_{\infty,2}^{s,[\frac{s}{2}]}(Q_T)$ we denote the space of function $f(x,t)$ which has derivatives $D_t^r D_x^k f(x,t) \in L^\infty(0,T;L^2(R))$ with $2r + k \leq s$.

We define the Fourier transform $F[f]$ and inverse Fourier transform $F^{-1}[f]$ for function $f(x)$ as follows

$$F[f] = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx, \quad F^{-1}[f] = \int_{-\infty}^{\infty} f(\xi) e^{2\pi i x \xi} d\xi$$