J. Partial Differential Equations
Vol. 5, *No.* 1 (1992), 43–56

AN ALGEBRAIC APPROACH FOR EXTENDING HAMILTONIAN OPERATORS*

Tu Guizhang and Ma Wenxiu**
(Computing Centre, Academia Sinica, Beijing 100080)
(Received Sept. 13, 1989; revised June 8, 1990)

Abstract An algebraic approach for extending Hamiltonian operators is proposed. A relevant sufficient condition for generating new Lie algebras from known ones is presented. Some special cases are discussed and several illustrative examples are given.
Key Words Matrix differential operator; Lie algebra; Hamiltonian operator.
Classification 58F05

1. Introduction

It is well known that many nonlinear evolution equations possess generalized Hamiltonian structures[1–4]. Hamiltonian operators play a crucial role in the algebraic and geometric theory of those Hamiltonian structures[5]. Based on Hamiltonian pairs, we can also construct, under certain conditions, a hierarchy of Hamiltonian equations possessing an infinite number of symmetries[6,7,8]. Therefore the search for new Hamiltonian operators and Hamiltonian pairs is one among the central topics in theory of Hamiltonian systems, there have been works[9,5,10] concerning the general theory of Hamiltonian operators. In the present paper, we propose an algebraic approach for extending Hamiltonian operators from lower orders to higher orders. We show that a large number of new Hamiltonian operators and new Hamiltonian pairs can be derived through this algebraic approach.

Let $u = (u_1(x,t), u_2(x,t), \cdots, u_q(x,t))$, $x, t \in R$, be a q-dimensional smooth function vector. The linear space of smooth functions $P[u] = P(x,t,u^{(m)}) = P(x,t,u,\cdots,u^{(m)})$, $m \geq 0$, is denoted by $\mathcal{A}$, $\mathcal{A}^q = \mathcal{A} \times \cdots \times \mathcal{A}(q \text{ times}) = \{(P_1, P_2, \cdots, P_q) | P_i \in \mathcal{A}, 1 \leq i \leq q\}$. Two functions P and Q of $\mathcal{A}$ are considered to be equivalent and denoted by $P \sim Q$ (mod D) if $P - Q = DR \equiv dR/dx$ holds for some $R \in \mathcal{A}$. The equivalent class that contains P is denoted by $\tilde{P} = \int P dx$, we call it a functional. The space of all functionals is represented by $\tilde{\mathcal{A}}$.

*The subject supported by the National Natural Science Foundation of China.
**Permanent Address: Institute of Mathematics, Fudan University, Shanghai 200433.

Definition 1 *A linear operator $J = J(x,t,u)$: $\tilde{\mathcal{A}}^q \to \tilde{\mathcal{A}}^q$ is called Hamiltonian if the bracket defined by*

$$\{\tilde{P},\tilde{Q}\} = \int \frac{\delta \tilde{P}}{\delta u}\Big(J\frac{\delta \tilde{Q}}{\delta u}\Big)^T dx, \quad \tilde{P},\tilde{Q} \in \tilde{\mathcal{A}}, \quad \frac{\delta}{\delta u} = \Big(\frac{\delta}{\delta u_1}, \frac{\delta}{\delta u_2}, \cdots, \frac{\delta}{\delta u_q}\Big) \tag{1.1}$$

is skew-symmetry

$$\{\tilde{P},\tilde{Q}\} = -\{\tilde{Q},\tilde{P}\}, \ \forall \tilde{P},\tilde{Q} \in \tilde{\mathcal{A}} \tag{1.2}$$

and satisfies the Jacobi identity

$$\{\{\tilde{P},\tilde{Q}\},\tilde{R}\} + \{\{\tilde{Q},\tilde{R}\},\tilde{P}\} + \{\{\tilde{R},\tilde{P}\},\tilde{Q}\} = 0, \quad \forall \tilde{P},\tilde{Q},\tilde{R} \in \tilde{\mathcal{A}} \tag{1.3}$$

In this case we call $\{\cdot,\cdot\}$ a Poisson bracket corresponding to the Hamiltonian operator J.

We observe that a matrix differential operator

$$J = (J_{ij})_{q\times q}, J_{ij} = \sum_{m=0}^{m(i,j)} P_m^{ij}[u] D^m, \quad D^m = \Big(\frac{d}{dx}\Big)^m, \quad P_m^{ij}[u] \in \mathcal{A} \tag{1.4}$$

may be considered as a linear operator $J : \mathcal{A}^q \to \mathcal{A}^q$, $P \mapsto JP^T$.

Definition 2[11] *If all the functions $P_m^{ij}[u], i,j = 1,2,\cdots,q, m = 0,1,\cdots,m(i,j)$, are linear with respect to u, then the operator J defined by (1.4) is called a u-linear operator; otherwise, J called a u-nonlinear operator.*

In this paper, we shall consider u-linear matrix differential operators with constant coefficients:

$$J = (J_{ij})_{q\times q}, \quad J_{ij} = \sum_{m=0}^{m(i,j)} \sum_{l=0}^{l(i,j)} \sum_{k=1}^{q} a_{ijlm}^k u_k^{(l)} D^m, \quad u_k^{(l)} = \Big(\frac{d}{dx}\Big)^l u_k \tag{1.5}$$

where the a_{ijlm}^k for all i,j,k,l,m are complex constants.

2. An Algebraic Approach

Let $J = J(u) : \mathcal{A}^q \to \mathcal{A}^q$ be a u-linear Hamiltonian operators as defined by (1.5) where $u = (u_1(x,t), u_2(x,t), \cdots, u_q(x,t))$. In the following we shall construct a new Hamiltonian operator $\bar{J} = \bar{J}(\bar{u}) : \bar{\mathcal{A}}^{qn} \to \bar{\mathcal{A}}^{qn}$, where $\bar{u} = (\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^n)$, $\bar{\mathcal{A}}^{qn} = \bar{\mathcal{A}}^q \times \cdots \times \bar{\mathcal{A}}^q$ (n times), and $\bar{u}^i = (u_{(i-1)q+1}(x,t), u_{(i-1)q+2}(x,t), \cdots, u_{iq}(x,t)), 1 \le i \le n$, $\bar{\mathcal{A}}^q = \bar{\mathcal{A}} \times \cdots \times \bar{\mathcal{A}}$ (q times) in which $\bar{\mathcal{A}}$ denotes the linear space of smooth functions $P[\bar{u}] = P(x,t,\bar{u}^{(m)}), m \ge 0$.