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# AN APPLICATION OF THE DUALITY METHOD TO THE REGULARITY OF MEMBRANE PROBLEMS WITH FRICTION

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**Abstract** Duality is applied to study the regularity of solutions to membrane problems with friction. The method consists of the characterization of the regularity of the subdifferentials of the friction functional. Then the regularity of a solution reduces to the regularity of a solution to a related elliptic boundary value problem or to that of an obstacle problem without friction.

**Key words** Regularity using duality; membrane problems with friction; subdifferentials.
**Classification** 35J85.

## 1. Introduction

We apply the duality method to obtain the regularity of solutions to membrane problems with friction, when the friction bound $g$ belongs to $L^p(\partial\Omega)$, $1 < p < \infty$, with mixed type boundary conditions. The regularity of membrane problem with friction has been considered under the assumption that $g$ belongs to $H^s(\partial\Omega)$, $-1/2 \leq s \leq 1/2$. We refer the reader to the survey paper by Lions [5]. More recently, Gasdaldi and Gilardi [4] considered the noncoercive case using the method of finite difference translation paralell to the boundary. Our present result is complementary to those mentioned above. It does not reproduce the optimal regularity when $g$ belongs to $H^s(\partial\Omega)$, however, it fills the gap that the previous authors did not cover.

The idea is based on an appropriate characterization of the subdifferential of the friction functional. Then the question of the regularity of a solution to the variational inequality is reduced to that of a solution to an appropriate elliptic equation or to that of a variational inequality without friction. We illustrate the idea as follows. Let $V$ be a Hilbert space and let $a : V \times V \to R$ be a continuous bilinear form. Assume that $j : V \to R$ is a proper and convex functional. Consider the variational inequality

$$\begin{cases} \text{find } u \in V \text{ such that } \forall v \in V \\ a(u, v-u) + j(v) - j(u) \geq (F, v-u) \end{cases} \tag{1.1}$$

Here $F \in V$ and $(\cdot,\cdot)$ is the inner product in $V$. Define the operator $A: V \to V$ such that

$$(Au, v) = a(u, v), \qquad \forall v \in V$$

Then $u \in V$ is a solution to (1.1) iff

$$j(v) - j(u) \geq (-Au + F, v - u), \qquad \forall v \in V \tag{1.2}$$

But then, by the definition of the subdifferential, $-Au + F \in \partial j(u)$. Therefore there exists a subgradient $u^* \in \partial j(u)$ such that

$$-Au + F = u^* \tag{1.3}$$

Once the regularity of $u^*$ is analyzed, the regularity of $u$ follows from that of $u^*$ and the regularity of the solutions to (1.3).

The membrane problem with friction, but without an obstacle, is considered in Section 2. The problem is described and set as a variational inequality. The regularity of the subgradients of the friction functional is given in Theorem 2.1. The regularity of the solution is then given in Theorem 2.2. The problem with friction and an obstacle is considered in Section 3 where similar results are obtained.

We remark that although we consider the membrane problem where $\Omega \subset R^2$, our results hold true in the general case $\Omega \subset \boldsymbol{R}^n$, with only minor modifications of the proof.

## 2. The Membrane with Friction

In this section we consider the membrane problem with friction in absence of any obstacle. Let $\Omega \subset R^2$ be a bounded domain, representing the vertical projection of the membrane, with boundary $\partial\Omega$ in $C^{1,1}$. Vertical forces of surface density $f: \Omega \to R$, act on the membrane. It is subject to contact with friction on a part $\Gamma_C$ of the boundary $\partial\Omega$, with friction bound $g: \Gamma_C \to R$, while on $\Gamma_D = \partial\Omega \setminus \overline{\Gamma}_C$ it is held fixed (possibly $\Gamma_D = \emptyset$). The classical formulation of the problem is ([2] or [5])

find $u: \Omega \to R$ such that

$$-\Delta u = f \quad \text{in } \Omega \tag{2.1}$$

$$u = 0 \qquad \text{on } \Gamma_D \tag{2.2}$$

$$\begin{cases} \left|\dfrac{\partial u}{\partial n}\right| \leq g \quad \text{on } \Gamma_C \text{ and} \\ \left|\dfrac{\partial u}{\partial n}\right| < g \quad \Rightarrow \quad u = 0 \\ \left|\dfrac{\partial u}{\partial n}\right| = g \Rightarrow \ u = -\lambda\dfrac{\partial u}{\partial n} \ \text{ for some } \lambda \geq 0 \end{cases} \tag{2.3}$$