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A FREE BOUNDARY PROBLEM GOVERNED BY NONLINEAR DEGENERATE EQUATIONS OF PARABOLIC TYPE *

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Abstract We consider a free boundary problem connected with non-Newtonian fluid motion, i.e. the flow of power law fluids with the yield stress. We obtain the solution of the relevant approximation problem by means of a parabolic quasi-variational inequality, and then obtain the weak solution of the original problem after a passage to the limit. Finally, we study the regularity of the weak solution.
Key Words Free boundary problems; nonlinear degenerate equations of parabolic type; quasi-variational inequality; non-Newtonian fluid.
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1. Introduction

One-dimensional motion of non-Newtonian fluids such as underground petroleum with high viscosity and high ester content and some other plastic fluids is governed by the modified Darcy's law[1] : $\vec{v} = \left[\frac{k}{\mu}(|p_x| - \tau)\right]^m, m > 1, |p_x| > \tau, \vec{v} = 0, |p_x| \le \tau$, where p is pressure distribution, τ is a positive constant related to the yield stress and rheological parameters, μ and k are positive constants. Under a suitable coordinate scale, $p_x \ge -\tau$, and by continuous equation, we obtain

$$p_t = (((p_x - \tau)^+)^m)_x, \quad m > 1$$

The examination of the motion behavior in semi-unbounded field $[0, \infty)$ needs giving correspondingly the initial distribution $p(x, 0) = p_0(x)$ and the pressure distribution $p(0, t) = g(t)$ at the end point $x = 0$. The discussion of the initial distribution $p_0'(x) - \tau > 0 \Leftrightarrow x \in [0, a), a > 0$ in this paper will lead to the following free boundary problems.

Denote $S_T = (0, \infty) \times (0, T), V(a) = \{v \in C^0([0, T]) \cap C^{0+1}([0, T] \setminus t_0), t_0 \in [0, T), v(0) = a, v' \ge 0,\ a.e.\ \}, \Omega(\lambda) = \{(x, t) : 0 < x < \lambda(t), 0 < t < T\}, E(\lambda) = \{u \in C^{2,1}(\Omega(\lambda)) \cap W^{1,1}_\infty(S_T), u_x \in C^0(\bar{\Omega}(\lambda)) \cap C^0(\overline{S_T \setminus \Omega(\lambda)}).$

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Problem I Find $\lambda(t) \in V(a)$, $p(x,t) \in E(\lambda)$ such that

$$p_t = ((p_x - \tau)^m)_x \qquad \text{in } \Omega(\lambda) \tag{1.1}$$

$$p|_{t=0} = p_0(x),\ p|_{x=0} = g(t) \tag{1.2}$$

$$p_x|_{x=\lambda(t)} = \tau,\ p|_{x=\lambda(t)} = p_0(\lambda(t)) \tag{1.3}$$

$$p_x - \tau > 0 \Leftrightarrow x \in [0, \lambda(t)) \tag{1.4}$$

Physically, the curve $x = \lambda(t)$ means disturbance front. Let us make the following hypotheses throughout this paper:

(H_1): $p_0(x) \in C^1([0,\infty))$, bounded, $(p_0'(x) - \tau)^m \in C^{0+1}([0,a])$ and $p_0'(x) - \tau > 0 \Leftrightarrow x \in [0,a)$,

(H_2): $g(t) \in C^{0+1}([0,T]), g'(t) \le 0$, a.e., $g(0) = p_0(0)$.

$\tau > 0, a > 0, m > 1$ in this paper are all known constants, and we shall introduce the constant

$$l = \frac{1}{\tau}\Big(\max_{[0,\infty)} |p_0(x)| + \max_{[0,T]} |g(t)|\Big) \tag{1.5}$$

Equation (1.1) is degenerate and we will examine its approximation problem firstly. By (H_1), there exists monotone decreasing sequence $\{\delta_j\}$ and monotone increasing sequence $\{a_{\delta_j}\}$ such that $\delta_j \downarrow 0$, $a_{\delta_j} \uparrow a$ and

$$p_0'(x) - \tau - \delta_j \begin{cases} > 0, & x < a_{\delta_j} \\ = 0, & x = a_{\delta_j} \\ < 0, & x > a_{\delta_j} \end{cases} \tag{1.6}$$

Problem I_δ: For $(\delta, a_\delta) \in \{\delta_j, a_{\delta_j}\}$, find $\lambda(t;\delta) \in V(a_\delta), p(x,t;\delta) \in E(\lambda(t;\delta))$ such that (1.1) in $\Omega_\delta = \Omega(\lambda(t;\delta))$, (1.2) and

$$p_x|_{x=\lambda(t;\delta)} = \tau + \delta, \quad p|_{x=\lambda(t;\delta)} = p_0(\lambda(t;\delta)) \tag{1.7}$$

$$p_x - \tau > \delta \Leftrightarrow x \in [0, \lambda(t;\delta)) \tag{1.8}$$

hold.

If $\lambda(t;\delta), p(x,t;\delta)$ are the solutions of I_δ, define a function

$$w(x,t;\delta) = \int_x^\infty (p_x(x,t;\delta) - \tau - \delta)^+ dx \tag{1.9}$$

(1.8) ensures $w > 0$ for $x < \lambda(t;\delta)$ and $w = 0$ for $x \ge \lambda(t;\delta)$. (1.9) gives $w(0,t;\delta) = p_0(\lambda(t;\delta)) - g(t) - (\tau+\delta)\lambda(t;\delta)$. Therefore we have $\lambda(t;\delta) < l$ and can easily verify that $\lambda(t;\delta), w(x,t;\delta)$ are the solutions of the following quasi-variational inequality.

Problem II_δ: Find $\lambda(t;\delta) \in V(a_\delta), \lambda(t;\delta) < l, w(x,t;\delta) \in W_q^{2,1}(Q_T), \forall q \ge 1, Q_T = (0,l) \times (0,T)$ such that

$$(((-w_x + \delta)^m)_x + w_t, v - w) \ge \Big(\frac{d}{dt}\psi_\delta(\lambda(t;\delta)), v - w\Big),$$