J. Partial Differential Equations
Vol. 5, *No.* 2 (1992), 94–96

A NOTE ON THE UNIQUENESS FOR DOUBLE DEGENERATE NONLINEAR PARABOLIC EQUATIONS

Yin Jingxue
(Department of Mathematics, Jilin University)

(Received July 23, 1990)

This note concerns with the initial-boundary value problem for double degenerate nonlinear parabolic equation

$$\frac{\partial \sigma(u)}{\partial t} = \sum_{k=1}^{n} \frac{\partial}{\partial x_i} A_i(\nabla u) - \varphi(u), \quad (x,t) \in Q_T \equiv \Omega \times (0,T) \tag{1}$$

where Ω is a bounded domain in R^n with smooth boundary, $\sigma(s), A_i(p_1, \cdots, p_n), \varphi(s)$ continuous functions with

$$\begin{gathered} (\sigma(s_1) - \sigma(s_2))(s_1 - s_2) > 0, \quad \forall s_1, s_2 \in R, \ s_1 \neq s_2 \\ \sum_{k=1}^{n} (A_i(p_1, \cdots, p_n) - A_i(q_1, \cdots, q_n))(p_i - q_i) \geq 0 \\ \forall p_j, q_j \in R, \quad j = 1, \cdots, n \\ (\varphi(s_1) - \varphi(s_2))(s_1 - s_2) \geq 0, \quad \forall s_1, s_2 \in R \end{gathered} \tag{H}$$

A typical example of (1) is the one in which

$$\sigma(s) = |s|^{q-2}s \ (q \geq 2), \quad A_i(p_1 \cdots, p_n) = |p_i|^{p-2}p_i \ (p \geq 2)$$
$$\varphi(s) = \lambda |s|^{\alpha-1}s \quad (\lambda \geq 0, \alpha > 0)$$

and the uniqueness with $n = 1$ $(\Omega = (0,1))$ was established in [1] for solutions satisfying

$$u \in L^\infty(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(0,T; L^p(\Omega))$$
$$\frac{d}{dt}(|u|^{(q-2)/2}u) \in L^2(0,T; L^2(\Omega))$$
$$\frac{d}{dt}(|u|^{q-2}u) \in L^{p'}(0,T; W^{-1,p'}(\Omega)), \quad \frac{1}{p} + \frac{1}{p'} = 1$$

While for any n the uniqueness was investigated in [2] for mild solutions in the slow diffusion case, namely for (1) in which $0 < q < 1, p > 2$.

In this note, we point out that under the much general condition (H) the uniqueness is also valid for generalized solutions in the sence of the following.

Definition *A function $u \in L^\infty(Q_T)$ is said to be a generalized solution of the initial-boundary value problem for (1) with initial data $u_0(x)$, if there exists $p > 1$ such that $u \in L^p(0,T;W_0^{1,p}(\Omega)), A_i(\nabla u) \in L^{p'}(Q_T)$ with $\frac{1}{p}+\frac{1}{p'}=1$ and for any $\psi(x,t) \in C^\infty(\bar{Q}_T)$ with $\psi(x,t)=0$ for $x \in \partial\Omega, t \in (0,T)$ or $x \in \bar{\Omega}, t = T$,*

$$\iint_{Q_T} \Big[\sigma(u)\frac{\partial \psi}{\partial t} - \sum_{k=1}^{n} A_i(\nabla u)\frac{\partial \psi}{\partial x_i} - \varphi(u)\psi\Big]dxdt + \int_\Omega \sigma(u_0)\varphi(x,0)dx = 0$$

Theorem *Let $\sigma(u_0) \in L^1(\Omega)$. Then the initial-boundary value problem for (1) has at most one generalized solution with $\frac{\partial \sigma(u)}{\partial t}$ being a finite regular measure on Q_T.*

Since the discussion is devoted to Equation (1) for multi-dimensional case under the much general assumption (H) in which $A_i(p_1,\cdots,p_n)$ may identically equal zero, our result supplements and generalizes those stated both in [1] and in [2].

The main idea of the proof follows basically from the one given in [3] and hence we only state the sketch of the proof and omit the details. The key step of the proof is to show the following inequality

$$\begin{aligned} J^+(u_1,u_2,\psi) \equiv \iint_{Q_T} H(u_1-u_2)\Big[&(\sigma(u_1)-\sigma(u_2))\frac{\partial \psi}{\partial t} - \sum_{k=1}^{n}(A_i(\nabla u_1) \\ &- A_i(\nabla u_2))\frac{\partial \psi}{\partial x_i} - (\varphi(u_1)-\varphi(u_2))\psi\Big]dxdt \geq 0 \end{aligned} \tag{2}$$

where u_1, u_2 are generalized solutions of Equation (1), $0 \leq \psi \in C_0^\infty(Q_T)$ and $H(s)=1$ for $s>0$ and $H(s)=0$ for $s \leq 0$. To do this, we introduce an approximate sequence $\{H_\varepsilon(s)\}$ satisfying

$$0 \leq H_\varepsilon(s) \leq 1, \quad 0 \leq sH_s'(s) \leq 1$$

$$\lim_{\varepsilon\to 0} H_\varepsilon(s) = H(s), \quad \lim_{\varepsilon \to 0} sH_\varepsilon'(s) = 0$$

Denote by $\tilde{u}_1, \tilde{u}_2$ the symmetric mean values of $u_1(\cdot,t), u_2(\cdot,t)$ as functions of t and consider the approximate functionals

$$\begin{aligned} J_\varepsilon^+(u_1,u_2,\psi) \equiv \iint_{Q_T} H_\varepsilon(\tilde{u}_1-\tilde{u}_2)\Big[&(\sigma(u_1)-\sigma(u_2))\frac{\partial \psi}{\partial t} - \sum_{k=1}^{n}(A_i(\nabla u_1) \\ &- A_i(\nabla u_2))\frac{\partial \psi}{\partial x_i} - (\varphi(u_1)-\varphi(u_2))\psi\Big]dxdt \geq 0 \end{aligned}$$

Proceeding similar to [3] and analysing carefully the right hand side of the above equality, we can obtain

$$J_\varepsilon^+(u_1,u_2,\psi) \geq \iint_{Q_T} H_\varepsilon(\tilde{u}_1-\tilde{u}_2)\frac{\partial}{\partial t}[\psi(\sigma(u_1)-\sigma(u_2))]$$