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NASH POINT EQUILIBRIA IN THE CALCULUS OF VARIATIONS*

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Abstract In this paper, the theory of Nash point equilibria for variational functionals including the following topics: existence in convex and non-convex cases, the applications to P. D. E., and the partial regularity, is studied. In the non-convex case, for a class of functionals, it is shown that the non-trivial solutions of the related systems of Euler equations are exactly the local Nash point equilibria and the trivial solution can not be a Nash point equilibrium.

Key Words Ky Fan's inequality; quasiconvexity; degree; Caccioppoli's inequality

Classification 35J, 49A.

0. Introduction

The theory of existence and regularity for the minima of variational integrals has been extensively developed. It leads to important results, namely, that the nonlinear elliptic systems of equations with variational structure, which arise from the calculus of variations in expressing the stationary conditions, have partial regular solutions, whereas this is false for general nonlinear elliptic systems.

This paper is to study Nash point equilibria (*Nashpoint* for short in the following) for variational integrals and extend the previous work on minima. As a result, a new class of nonlinear elliptic systems which stands between the variational case and the case of general nonlinear elliptic systems and consists of a system of Euler equations of a Nashpoint problem for several variational integrals (see (0.9) below), can be solved via this approach (see Remark 2.2). The concept of Nashpoints is from Game Theory [1], [13].

Definition 0.1 *Let* $E_1, E_2, \cdots, E_m$ *be* m *sets and* $J_1, J_2, \cdots, J_m$ *be* m *functions from* $E = \prod_{i=1}^{m} E_i$ *to* $\boldsymbol{R}$. *A point* $u = (u_1, u_2, \cdots, u_m) \in E$ *is a Nashpoint for* $J_1, J_2, \cdots, J_m$

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on E if for every $v_i \in E_i, i = 1, \cdots, m$,

$$\begin{aligned} J_1(u_1, u_2, \cdots, u_m) &\le J_1(v_1, u_2, \cdots, u_m) \\ J_2(u_1, u_2, \cdots, u_m) &\le J_2(u_1, v_2, \cdots, u_m) \\ \cdots \quad & \quad \cdots \\ J_m(u_1, u_2, \cdots, u_m) &\le J_m(u_1, u_2, \cdots, v_m) \end{aligned} \tag{0.1}$$

The concept of Nashpoints is a natural extension of minimum points ($m = 1, J_1 = J$) and of saddle points ($m = 2, J_1 = J, J_2 = -J$). The function $\varphi : E \times E \to \boldsymbol{R}$ induced by the functions $J_1, J_2, \cdots, J_m$ is quite useful, which is defined by

$$\varphi(u, v) = \sum_{i=1}^{m} [J_i(u_1, u_2, \cdots, u_m) - J_i(u_1, \cdots, u_{i-1}, v_i, u_{i+1}, \cdots, u_m)] \tag{0.2}$$

Then, $u \in E$ is a Nashpoint for the functions $J_1, J_2, \cdots, J_m$ if and only if $u \in E$ and

$$\varphi(u, v) \le 0, \qquad \forall v \in E \tag{0.3}$$

In order to simplify the notations and without loss of generality, we only consider the case $m = 2$. The setting of our study is as follows.

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded open subset and let f and $g : \Omega \times \boldsymbol{R}^{N_1} \times \boldsymbol{R}^{N_2} \times \boldsymbol{R}^{nN_1} \times \boldsymbol{R}^{nN_2} \to \boldsymbol{R}$ be two Carathéodory functions. We have two integral functionals

$$J(u, v) = J(u, v; \Omega) = \int_\Omega f(x, u(x), v(x), Du(x), Dv(x))dx \tag{0.4}$$

$$K(u, v) = K(u, v; \Omega) = \int_\Omega g(x, u(x), v(x), Du(x), Dv(x))dx \tag{0.5}$$

for $u \in W^{1,p}(\Omega, \boldsymbol{R}^{N_1})$ and $v \in W^{1,p}(\Omega, \boldsymbol{R}^{N_2})$. For simplicity, we assume $N_1 = N_2 = N$ in the following. There is no basic simplification if $N_1 = N_2 = N$. In this paper, a point $s \in \boldsymbol{R}^{2N} = \boldsymbol{R}^N \times \boldsymbol{R}^N$ or a point $\xi \in \boldsymbol{R}^{2nN} = \boldsymbol{R}^{nN} \times \boldsymbol{R}^{nN}$ is always written in the following splitted form, when necessary,

$$s = (s_1, s_2), \quad \text{where } s_1 \text{ and } s_2 \in \boldsymbol{R}^N$$

or

$$\xi = (\xi_1, \xi_2), \quad \text{where } \xi_1 \text{ and } \xi_2 \in \boldsymbol{R}^{nN}$$

According to (0.2), we have the induced function

$$\varphi(U, U') = \varphi(U, U'; \Omega) = J(u, v; \Omega) - J(u', v; \Omega) + K(u, v; \Omega) - K(u, v'; \Omega) \tag{0.6}$$

for $U = (u, v)$ and $U' = (u', v')$ in $W^{1,p}(\Omega, \boldsymbol{R}^{2N})$. We define a function h induced by f and g as

$$h(x, s, \bar{s}, \xi, \bar{\xi}) = f(x, s_1, s_2, \xi_1, \xi_2) - f(x, \bar{s}_1, s_2, \bar{\xi}_1, \xi_2) + g(x, s_1, s_2, \xi_1, \xi_2) - g(x, s_1, \bar{s}_2, \xi_1, \bar{\xi}_2) \tag{0.7}$$