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A NOTE ON $C^{1,\alpha}$ ESTIMATES FOR SOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS AND OBSTACLE PROBLEMS*

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Abstract We deal with $C^{1,\alpha}$ interior estimates for solutions of fully nonlinear equation $F(D^2u, Du, x) = f(x)$ with the bounded gradient Du and a bounded $f(x)$. Based on these estimates we obtain the existence of strong solutions of the obstacle problem for fully nonlinear elliptic equations under natural structure conditions.
Key Words Hölder estimate for the gradient; viscosity solutions; mollification approach; obstacle problems
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1. Introduction

In this note, we deal with $C^{1,\alpha}$ interior estimates for solutions of fully nonlinear elliptic equations

$$F(D^2u, Du, x) = f(x) \tag{1.1}$$

with the bounded gradient Du and a bounded $f(x)$. In [1], L.A.Caffarelli obtains $C^{1,\alpha}$ interior estimates for solutions of

$$F(D^2u, x) = f(x)$$

with $F(r, x)$ continuous with respect to x. Obviously the result cannot be used in the above case. If $Df(x)$ is of $L^n(\Omega, R^n)$, we can get $C^{1,\alpha}$ estimate for solutions with some $\alpha \in (0,1)$ after differentiating the equation and using Hölder estimates for Du, but it does not work for only bounded $f(x)$. For quasilinear elliptic equation

$$a^{ij}(Du, x)D_{ij}u = f(x)$$

the similar results as shown in this note have been obtained (cf Theorem 13.6 in [3]) because certain combinations of Du satisfy the equations of divergence form and the De Giorgi-Nash-Moser estimates can be used. The method is not adaptable to fully nonlinear equations.

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Our method is based on the modification of the approximation lemma given by L.A.Caffarelli (Lemma 13 in [1]), but some differences can be found in the proofs. In addition we also show how to use the mollification approach to get $C^{1,\alpha}$ estimates instead of the iteration approach as shown in [1].

Let Ω be an open bounded domain in R^n and $\mathcal{S}^n(\mathcal{S}^n_+)$ be the space consisting of (positive definite) symmetric matrices. Assume that $F(r,p,x)$ in $\Gamma = \mathcal{S}^n \times R^n \times \Omega$ satisfies the following structure conditions:

$$\lambda|s| \leq F(r+s,p,x) - F(r,p,x) \leq \Lambda|s|, \quad \forall s \in \mathcal{S}^n_+ \tag{1.2}$$

$$F(0,p,x) = 0 \tag{1.3}$$

$$|F(r,p,x+y) - F(r,p,x)| \leq \lambda\beta(y)(1+|r|+|p|), \quad \forall x+y \in \Omega \tag{1.4}$$

$$(1+|p|)|F(r,p+q,x) - F(r,p,x)| \leq \lambda\mu|q|(1+|r|+|p|), \quad \forall q \in R^n \tag{1.5}$$

for $(r,p,x) \in \Gamma$, where λ, Λ, μ are positive constants and $\beta(y) > 0$ in R^n.

Theorem 1.1 *Suppose $0 < \bar{\alpha} < 1$. Assume that solution w to the equation*

$$F(D^2 w, Dw, 0) = 0 \quad \text{in } B_R$$

satisfy the a priori estimate

$$\|w\|_{C^{1,\bar{\alpha}}(B_{R/2})} \leq C_0 R^{-(1+\bar{\alpha})}\|w\|_{L^\infty(B_R)} \tag{1.6}$$

For $0 < \alpha < \bar{\alpha}$, let $u \in C^{1,\alpha}(\Omega)$ be a viscosity solution of

$$F(D^2 u, Du, x) = f(x) \quad \text{in } \Omega \tag{1.7}$$

with $\|Du\|_{L^\infty(\Omega)} \leq M^*$. Assume that $F(r,p,x)$ satisfies (1.2)–(1.5) in Γ and

$$\sup_{\substack{x_0 \in \Omega \\ 0 \leq R \leq 1}} \left\{ R^{-\bar{\alpha}n} \int_{B_R(x_0)\cap\Omega} |f(x)|^n dx \right\}^{\frac{1}{n}} \leq \lambda F_0 \tag{1.8}$$

$$\omega(R) = \left\{ \fint_{B_R} \beta^n(x) dx \right\}^{\frac{1}{n}} \to 0 \quad \text{as } R \to 0 \tag{1.9}$$

Then there exists C depending on n, $\frac{\Lambda}{\lambda}$, μ, M^*, F_0, $\omega(\cdot)$, α, $\bar{\alpha}$ and C_0, such that

$$[Du]^*_{0,\alpha,\Omega} \leq C[\|Du\|_{L^\infty(\Omega)} + F_0 + 1] \tag{1.10}$$

where the semi-norm $[u]^*_{k,\alpha}$ is referred to [3] (p. 61).

As an application we obtain the $W^{2,\infty}$ strong solutions of the obstacle problem for fully nonlinear elliptic equations under natural structure conditions, which improves the results in [2] and [4].