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CAUCHY PROBLEM FOR A GENERALIZED NONLINEAR DISPERSIVE EQUATION

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Abstract The solvability of global smooth solution for the Cauchy problem of a generalized nonlinear dispersive equation is studied by using the continuation method. In addition, the convergences of solution for this problem are also discussed.
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1. Introduction and Main Results

We are concerned with the existence uniqueness and convergence of solution for the following Cauchy problem

$$\partial_t u + \beta\partial_x^3 u + \delta\partial_x^2 H(u) + \gamma\partial_x H(u) + \partial_x g(u) = 0 \tag{1.1}$$

$$u(x,0) = \varphi(x) \tag{1.2}$$

for $x \in R$, $t \in R^+$, $k \in N$, where β, δ, γ are real constants, $H(u) = \frac{1}{\pi}P.\int_{-\infty}^{\infty}\frac{u(y,t)}{y-x}dy$ is the Hilbert transform, $P.$ denotes the Cauchy principal value. It is well known that Eq. (1.1) plays an important rule in many branches of physics. For instance, if $g(u) = cu + u^2$ and $\gamma = 0$, then Eq. (1.1) reduces to Benjamin-Ono-KdV equation[1]

$$\partial_t u + \beta\partial_x^3 u + \delta\partial_x^2 H(u) + \partial_x g(u) = 0 \tag{1.3}$$

which describes a large class of internal waves in the ocean and the stratified fluid. Moreover, if $\delta = 0$ or $\beta = 0$, $c = 0$, then the BO-KdV equation (1.3) reduces respectively to the well known Korteweg-de Vries equation[2] and Benjamin-Ono equation[3], namely

$$\partial_t u + \beta\partial_x^3 u + \partial_x g(u) = 0 \tag{1.4}$$

$$\partial_t u + \delta\partial_x^2 H(u) + u\partial_x u = 0 \tag{1.5}$$

which describes respectively the propagation of long waves of finite amplitude and the internal waves in deep water. Next, when $\delta = 0$, Eq. (1.1) reduces to the following equation

$$\partial_t u + \beta\partial_x^3 u + \gamma\partial_x H(u) + \partial_x g(u) = 0 \tag{1.6}$$

which is the general equation obtained by Ott and Sudan[4] in studying ionacoustic waves of finite amplitude with the linear Landau damping.

The Cauchy problems for KdV equation (1.4) have been studied very well. For instance, Bona and Smith[5], Zhou and Guo[6] and their references. Recently, the existence of a unique global solution for the Cauchy problem of BO equation (1.5) ($\delta = 1$) has been proved by Refael[7] with the semi-group method and by Zhou[8] with the continuation method.

So far the solvability of Cauchy problem (1.2) for the nonlinear dispersive equations (1.3) and (1.6) are not yet found to be discussed. In the present paper, we are concerned with the Cauchy problem (1.2) for the generalized nonlinear dispersive equation (1.1). Indeed, under some conditions on the function $g(u)$, first we demonstrate the existence and uniqueness of solution for the Cauchy problem (1.2) of the following nonlinear diffusion equation

$$\partial_t u + \varepsilon \partial_x^4 u + \beta \partial_x^3 u + \delta \partial_x^2 H(u) + \gamma \partial_x H(u) + \partial_x g(u) = 0 \qquad (1.7)$$

with $\varepsilon > 0$. The solvability of global smooth solution for problem (1.1) (1.2) is approximated by the solution of problem (1.7) (1.2) as the small coefficient ε tends to zero. In addition, the convergences of solution for problem (1.1) (1.2) as $\delta \to 0$ and $\gamma \to 0$ are also discussed

Throughout the paper we use the following notations:

$$\partial_t = \frac{\partial}{\partial t}, \partial_x = \frac{\partial}{\partial_x}, Q_T = [0,T] \times R, R = (-\infty, \infty) \text{and } T > 0 \text{ is a constant;}$$

$L^p(I;B)$ denotes the space of measurable functions from $I = [0,T]$ to a Banach space B, such that $\|u(\cdot)\|_B \in L^p(I)$, $1 \le p \le \infty$;

$$W_p^{k,[\frac{k}{4}]} = \{u | u \in L^p(Q_T), \partial_x^r \partial_t^s u(x,t) \in L^p(Q_T) \text{ for } 1 \le p \le \infty, 4s + r \le k\};$$

$$W_\infty^r(0,T;H^{s-nr}(R)) = \Big\{u | \partial_t^l u(x,t) \in L^\infty(0,T;H^{s-nl}(R)) \text{ for } l = 0,1,2,\cdots,r$$

and

$$r \le \left[\frac{s}{n}\right]\Big\};$$

$$\|u\|_p = \|u(x)\|_{L^p(R)}; \int_{-\infty}^{\infty} \Phi(x)dx = \int \Phi$$

Different positive constants might be denoted by the same letter C. If necessary, by $C(*,*)$ we denote the constant depending only on the quantities appearing in parentheses.

With these notations we state our main results:

Theorem 1 *Let $\varepsilon, \beta, \delta, \gamma$ be real constants satisfying $\varepsilon > 0$, and the initial function $\varphi(x) \in H^k(R)$, and $g(\cdot) \in C^k(R)$ $(k \ge 2)$. If there exists a constant C such that*

$$|g(v)| \le C(|\nu| + |\nu|^7)$$