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# ON THE EIGEN-VALUE PROBLEM FOR ELLIPTIC SYSTEMS UNDER NATURAL GROWTH CONDITION: THE SUPER LIMIT CASES

Xu Haixiang
(Dept. of Math., Hunan University, Changsha, 410082 )
Yang Linge
(Dept. of Math. and Mech., Cent. South Univ. of Tech., Changsha, 410083)

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**Abstract** In this paper, we consider a quasilinear elliptic eigenvalue problem. For the unbounded coefficients, we allow the eigen exponent greater than the Sobolev imbedding critical exponent; for the equations, we also allow the coefficient of suitable smallness condition greater than 1.

**Key Words** Natural growth condition; super Sobolev imbedding exponent; super suitable smallness condition.

**Classifications** 35J, 35P.

## 1. Introduction

Let $\Omega$ be a bounded domain in $R^n$ with suitable smoothly boundary $\partial\Omega$. We consider the following eigenvalue problem for the elliptic systems under natural growth conditions

$$(\mathrm{I})\begin{cases} -D_\alpha([u]^{p-2}a^{\alpha\beta}(x,u)D_\beta u^i)+(1/2)[u]^{p-2}D_{u^i}a^{\alpha\beta}(x,u)D_\alpha u^j D_\beta u^j=\lambda|u|^{q-2}u^i \\ \qquad\qquad\qquad\qquad\qquad\qquad i=1,\cdots,N;\ x\in\Omega \\ u=0 \qquad\qquad x\in\partial\Omega \end{cases}$$

where $[u]=(a^{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i)^{1/2}$. By convention, repeated Greek indices are summed from 1 to $n$, Latin indices from 1 to $N$. Unless otherwise stated all integrals are taken over $\Omega$.

Problem (I) may be regarded as the Euler equations of the functional

$$I(u)=(1/p)\int[u]^p dx \tag{1}$$

under the constrain

$$K_1=\{u\in W_0^{1,p}(\Omega),\ \|u\|_q=1\} \tag{2}$$

The research for this problem was started in Ref [1]. In Ref.[1], $N = 1, q < p^* = np/(n-p), n > p = 2, (a^{\alpha\beta})$ is a uniformly bounded and strictly positive definite symmetric matrix.

In Ref.[2], Shen Yaotian and his coleagues broke the restriction of uniformly bounded coefficient and so on, they considered the problem (I) when $p = 2, q < p^*$ and the strong nonlinear natural growth which satisfies the $\Delta_2$ condition. The method used in Refs.[1], [2] was to consider the minimization problem of the functional (1) under the constrain $K_1$, and they proved that for some special admissible functions the minimum $u$ satisfies the Euler equations. Furthermore by the De Giorgi-Moser techniques they proved the boundness of $u$.

In Ref. [3], the author considered the problem (I) under the general natural growth condition when $q < p^*$. Different from Refs. [1] [2], we consider the minimization problem of the functional (1) under the constrain

$$K_2 = \{u \in W_0^{1,p}(\Omega),\ \|u\|_q = 1,\ \sup_\Omega |u| \le K\}$$

then we prove the minimum $u$ of abov› problem satisfies the variational inequality for some special admissible functions, and by the Moser iteration we obtain a uniformly prior estimate for the $L^\infty$ norm of the minimum $u$, so $u$ satisfies the Euler equations.

The Refs. [1]–[3] have a common suitable smallness condition: there exists a constant $0 < a < 1$, such that

$$(-1/2)u^i D_{u^i} a^{\alpha\beta}(x,u)\xi_\alpha\xi_\beta \le a a^{\alpha\beta}(x,u)\xi_\alpha\xi_\beta$$
$$\text{for a.e.} \quad (x,u,\xi) \in \Omega \times R^N \times R^n \tag{3}$$

Related to the regularity theory of elliptic systems [4], the condition $0 < a < 1$ in (3) is reasonable in some sense, but if we pay attention to the work of Ladyzenskaya and Ural'tseva [5] about the Hölder continuity of the bounded weak solution for elliptic equations under the natural growth condition ($N = 1$), we may ask the question: does there exist a solution in the super limit case for the suitable smallness condition ($N = 1$)? In this paper, we attempt to prove the existence of problem (I) in some super limit cases for the suitable smallness condition when $N = 1$.

Another question is that, for the natural growth condition itself there is no growth restriction about $u$, so if the eigen exponent $q$ is equal to or greater than the critical exponent of Sobolev imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ $(q \ge p^*)$, does there also exist a solution of problem (I)? In this case, in addition to the instinctive difficulty of the natural growth functional (it means the differentiability of the natural growth functional), we also have the difficulty with lack of compactness $(q \ge p^*)$ or boundness $(q > p^*)$. In this paper, by using a variation [3] of Hildebrandt's variational method [6], we present an existence result of problem (I) for some super Sobolev imbedding critical exponent when the coefficient is unbounded.