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THE STEFAN PROBLEM WITH NONLINEAR CONVECTION *

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Abstract In this paper, we consider the time dependent Stefan problem with convection in the fluid phase governed by the Navier-Stokes equation and with adherence of the fluid on the lateral boundaries. The existence of a weak solution is obtained *via* the introduction of a temperature dependent penalty term in the fluid flow equation, together with the application of various compactness arguments.

Key Words Stefan problem; nonlinear convection; penalty method; compactness arguments.

Classifications 35Q10; 35R35.

1. Introduction

J. R. Cannon, E. DiBenedetto and G. H. Knightly discussed the bidimensional Stefan problem with convections in paper [1], the convection is governed by a linear Stokes equation there. In this paper we consider the Stefan problem with nonlinear convection, i.e., the convection is governed by a Navier-Stokes equation

$$\frac{\partial \vec{V}}{\partial t} - \nu \triangle \vec{V} + (\vec{V} \cdot \nabla)\vec{V} + \nabla_x p = \vec{f}(u) \text{ in } \Omega_1 \tag{1.1}$$

where Ω_1 is a domain of liquid phase in $Q_T = \Omega \times (0, T]$, $\Omega \subset R^2$ is a bounded domain. Let $\Omega_2 = Q_T \backslash \bar{\Omega}_1$ denote a domain of solid phase, $\vec{V} = \vec{V}(x,t)$ denote the velocity of liquid flow, $u = u(x,t)$ denote the quantity determined by the temperature (see [1]), where $x = (x_1, x_2)$, $t \in [0,T]$. Let $\Omega(t) = \Omega \times \{t\}, t \in [0,T]$.

As in [1], the problem reduces to (1.1)–(1.6)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle K(u) + \vec{V} \cdot \nabla_x u = 0, & (x,t) \in Q_T \\ -\nabla_x K(u) \cdot \vec{N} = g(x,t,u), & (x,t) \in S_T \\ u(x,0) = u_0(x), & x \in \Omega(0) \\ u(x,t) = 0, & (x,t) \in \Gamma \\ \{[\nabla_x K(u)]^+ - [\nabla_x K(u)]^-\} \cdot \nabla_x \Phi = L\Phi_t, & (x,t) \in \Gamma \end{cases} \tag{1.2}$$

where $S_T = \bigcup_{0<t\leq T} \partial\Omega(t)$ is the lateral boundary of Q_T and $\Gamma \equiv \Gamma_T \equiv \bigcup_{0\leq t\leq T} \Gamma(t)$ is the free boundary, while $\Gamma(t)$ determined by $\phi(x,t) = 0$. We set $S_i = \bar{\Omega}_i \bigcap \bar{S}_T$ $(i = 1, 2)$.

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$\vec{V}$ satisfies conditions

$$\vec{V} = 0 \quad \text{on } S_1 \tag{1.3}$$

$$\vec{V}(x,0) = \vec{V}_0(x) \quad x \in \Omega_1(0), \ \text{div}\vec{V}_0(x) = 0 \tag{1.4}$$

$$\text{div}\vec{V} = 0 \tag{1.5}$$

$$\vec{V}(x,t) = 0 \quad \text{a.e. on } \Omega_2 \tag{1.6}$$

Briefly, if there is no additional description, the notations are similar to those described in paper [1].

2. The Equations of the Weak Formulation

Let $\phi \in W_2^{1,1}(Q_T)$ such that $\phi(x,T) = 0$. Then multiplying the first equation of (1.2) by ϕ and performing integration by parts, we obtain (note: $\nabla_x u\phi = (\nabla_x u)\phi$)

$$\iint_{Q_T} \{ -\beta(u)\phi_t + \nabla_x K(u) \cdot \nabla_x \phi + \vec{V} \cdot \nabla_x u\phi\} dxdt$$

$$= -\int_{S_T} g(x,t,u)\phi ds + \int_{\Omega(0)} \beta(u_0)\phi(x,0)dx \tag{2.1}$$

where $\beta(\cdot)$ is the maximal monotone graph

$$\beta(s) = \begin{cases} s, & \text{for } s > 0 \\ [-L, 0], & \text{for } s = 0 \\ s - L, & \text{for } s < 0 \end{cases} \tag{2.2}$$

Since the graph $\beta(\cdot)$ is multivalued, $\beta(u(x,t))$ has to be interpreted as a function $w(x,t) \subset \beta(u(x,t))$, the inclusion being intended in the sense of the graphs. In order to simplify the symbolism we will keep the symbol $\beta(u(x,t))$, bearing in mind the way it has to be interpreted. Since $u_0 \neq 0$ except on $\Gamma, \beta(u_0(x))$ is unambiguously a.e. defined in $\Omega(0)$.

To obtain a weak formulation of (1.1), (1.3)–(1.6), consider a smooth, divergence free vector value function $\vec{\psi}$ which is compactly supported in $\Omega_1(t) = \Omega_1 \bigcap \{t = t\}$, for all $t \in [0,T]$ and $\vec{\psi}(x,T) \equiv 0$. Take the "dot" product of (1.1) by $\vec{\psi}$ and integrate by parts in Ω_1. Routine calculations give

$$\iint_{\Omega_1} \{ -\vec{V} \cdot \vec{\psi}_t + \nu \nabla_x \vec{V} : \nabla_x \vec{\psi} + (\vec{V} \cdot \nabla)\vec{V} \cdot \vec{\psi} - \vec{f}(u)\vec{\psi}\} dxdt$$

$$= \int_{\Omega_1(0)} \vec{V}_0(x)\vec{\psi}(x,0)dx \tag{2.3}$$

where $\Omega_1 = \{(x,t) \in Q_T | u(x,t) > 0\}$.

Definition *By a weak solution of* (1.1)–(1.6), *we mean a pair* $(u, \vec{V})$ *such that*

(1) $\vec{V} \in V_2(Q_T)$, $u \in V_2(Q_T) \bigcap C(Q_T)$;

(2) $\vec{V} \in J_1^\infty(Q_T) = L^\infty(0,T; J(\Omega)) \bigcap L^2(0,t; J_1(\Omega))$,