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THE PROBLEM OF EIGENVALUE ON NONCOMPACT COMPLETE RIEMANNIAN MANIFOLD

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Abstract Let M be an n-dimensional noncompact complete Riemannian manifold, "Δ" is the Laplacian of M. It is a negative selfadjoint operator in $L^2(M)$.

First, we give a criterion of non-existence of eigenvalue by the heat kernel. Applying the criterion yields that the Laplacian on noncompact constant curvature space form has no eigenvalue.

Then, we give a geometric condition of M under which the Laplacian of M has eigenvalues. It implies that changing the metric on a compact domain of constant negative curvature space form may yield eigenvalues.
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1. Introduction

If M is a compact Riemannian manifold, then the Laplacian of M has pure point spectrum. If M is noncompact, the Laplacian may have essential spectrum. For example, the Laplacian of R^n has purely continuous spectrum [6]. It is interesting that, under what geometric conditions, the Laplacian of M always has or never has eigenvalue ([10], Appendix 1).

H. Donnelly, P. Li ([2], [3], [4]), J. F. Escobar ([5]) considered the problem and mainly proved that

(1) The Laplacian of constant negative curvature space form has no eigenvalue.

(2) If M is simply connected and noncompact with a rotational invariant metric of non-negative curvature, then the Laplacian of M has no eigenvalue.

In another paper, the author proved that, if M is a complete noncompact manifold with nonnegative Ricci curvature which possess a pole the essential spectrum of the Laplacian is $(-\infty, 0]$.

In this paper, we give a criterion for complete Riemannian manifold to have purely continuous spectrum by the heat kernel of M. By the criterion, we give a simple proof the Laplacian of R^n and the Laplacian of constant negative curvature space form have no eigenvaiue.

In [9], Cheng J. C. and the author proved that there exist simply connected strongly negatively curved manifolds, on which the Laplacians have eigenvalues. In this paper, we give a geometric condition under which the Laplacian has eigenvalues. The result shows that although the Laplacian of a constant negative curvature space form has no eigenvalue, changing the metric on a compact domain may yield eigenvalues.

2. A Criterion of Non-existence of Eigenvalue

Theorem 2.1 *Let M be a complete Riemannian manifold, $H_t(x,y)$ is the heat kernel of $M, \lambda > 0$. If*

$$\lim_{N\to\infty} \frac{e^{2N}}{\lambda^{2N}} \left| \frac{\partial^{2N} H_t(x,y)}{\partial t^{2N}} \right|_{t=\frac{2N}{\lambda}} = 0 \quad \text{for all } x \in M \tag{2.1}$$

then $-\lambda$ is not an eigenvalue of the Laplacian of M.

Proof Suppose $-\lambda$ is an eigenvalue of Δ, u is the eigenfunction.

Set $u(x,t) = \int_M H_t(x,y)u(y)dy, \quad w(x,t) = e^{-\lambda t}u(x)$.

Clearly,

$$\left(\Delta - \frac{\partial}{\partial t}\right)u(x,t) = 0, \; u(x,0) = u(x)$$

$$\left(\Delta - \frac{\partial}{\partial t}\right)w(x,t) = 0, \; w(x,0) = u(x)$$

Applying the uniqueness of L^2-solution for heat equation, we have

$$e^{-\lambda t}u(x) = \int_M H_t(x,y)u(y)dy$$

Differentiating the last equality with respact to t, we have

$$(-\lambda)^N e^{-\lambda t}u(x) = \int_M \frac{\partial^N H_t(x,y)}{\partial t^N} u(y)dy$$

Hence,

$$u^2(x) \le \frac{e^{2\lambda t}}{\lambda^{2N}} \int_M \left(\frac{\partial^N H_t(x,y)}{\partial t^N}\right)^2 dy \int_M u^2(y)dy \tag{2.2}$$

On the other hand, $H_{2t}(x,x) = \int_M H_t^2(x,y)dy$

$$\frac{\partial^{2N} H_{2t}(x,x)}{\partial t^{2N}} = 2^{2N} \int_M \frac{\partial^{2N} H_t(x,y)}{\partial t^{2N}} H_t(x,y)dy = 2^{2N} \int_M \Delta^{2N} H_t(x,y) H_t(x,y)dy$$

$$= 2^{2N} \int_M \Delta^N H_t(x,y) \cdot \Delta^N H_t(x,y)dy = 2^{2N} \int_M \left(\frac{\partial^N H_t(x,y)}{\partial t^N}\right)^2 dy$$

Hence,

$$\left.\frac{\partial^{2N} H_s(x,x)}{\partial s^{2N}}\right|_{s=2t} = \int_M \left(\frac{\partial^N H_t(x,y)}{\partial t^N}\right)^2 dy \tag{2.3}$$