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LIFE-SPAN OF CLASSICAL SOLUTIONS TO NONLINEAR WAVE EQUATIONS IN TWO-SPACE-DIMENSIONS II

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Abstract In two-space-dimensional case we get the sharp lower bound of the life-span of classical solutions to the Cauchy problem with small initial data for fully nonlinear wave equations of the form $\Box u = F(u, Du, D_x Du)$ in which $F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha})$ with $\alpha = 2$ in a neighbourhood of $\hat{\lambda} = 0$. The cases $\alpha = 1$ and $\alpha \geq 3$ have been considered respectively in [1] and [2].
Key Words Life-span; classical solution; Cauchy problem; nonlinear wave equation

Classification 35G25, 35L15, 35L70, 35L05

1. Introduction

Consider the Cauchy problem for fully nonlinear wave equations

$$\Box u = F(u, Du, D_x Du) \tag{1.1}$$

$$t = 0 : u = \varepsilon\phi(x), u_t = \varepsilon\psi(x) \tag{1.2}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{1.3}$$

is the wave operator,

$$D_x = \Big(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\Big), \quad D = \Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\Big) \tag{1.4}$$

$\phi, \psi \in C_0^\infty(R^n)$ and $\varepsilon > 0$ is a small parameter.

Let

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \cdots, n; (\lambda_{ij}), \ i, j = 0, 1, \cdots, n, i + j \geq 1) \tag{1.5}$$

Suppose that in a neighbourhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \leq 1$, the nonlinear term $F = F(\hat{\lambda})$ in (1.1) is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}) \tag{1.6}$$

where α is an integer ≥ 1.

Our aim is to study the life-span of classical solution to (1.1)–(1.2) for $n = 2$ and all integers $\alpha \geq 1$. By definition, the life-span $\tilde{T}(\varepsilon) = \sup \tau$ for all $\tau > 0$ such that there exists a classical solution to (1.1)–(1.2) on $0 \leq t \leq \tau$.

In the previous papers [1] and [2] we have respectively considered the cases $\alpha = 1$ and $\alpha \geq 3$. The result is the following:

$$\tilde{T}(\varepsilon) = +\infty \quad \text{if } n = 2 \text{ and } \alpha \geq 3 \tag{1.7}$$

while if $n = 2$ and $\alpha = 1$,

$$\tilde{T}(\varepsilon) \geq \begin{cases} be(\varepsilon) \\ b\varepsilon^{-1}, & \text{if } \int_{R^2} \psi(x)dx = 0 \\ b\varepsilon^{-2}, & \text{if } \partial_u^2 F(0,0,0) = 0 \end{cases} \tag{1.8}$$

where b is a positive constant and $e(\varepsilon)$ is defined by

$$\varepsilon^2 e^2(\varepsilon) \ln(1 + e(\varepsilon)) = 1 \tag{1.9}$$

In this paper we will consider the remainder case $n = 2$ and $\alpha = 2$ and prove

$$\tilde{T}(\varepsilon) \geq \begin{cases} b\varepsilon^{-6} \\ \exp\{a\varepsilon^{-2}\}, & \text{if } \partial_u^\beta F(0,0,0) = 0 \ (\beta = 3, 4) \end{cases} \tag{1.10}$$

where a, b are positive constants. For this purpose, some refined estimates are needed.

All results mentioned above are sharp due to H.Lindblad [3], Zhou Yi [4]–[5] etc.

In order to prove the desired result, by differentiation, is suffices to consider the Cauchy problem for the following general kind of quasilinear wave equations

$$\Box u = \sum_{i,j=1}^{2} b_{ij}(u, Du) u_{x_i x_j} + 2\sum_{j=1}^{2} a_{0j}(u, Du) u_{t x_j} + F_0(u, Du) \tag{1.11}$$

$$t = 0: \quad u = \varepsilon\phi(x), \quad u_t = \varepsilon\psi(x) \tag{1.12}$$

where $x = (x_1, x_2)$, $\Box u = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x_1^2} - \dfrac{\partial^2}{\partial x_2^2}$, $\varepsilon > 0$ is a small parameter,

$$\phi, \ \psi \in C_0^\infty(R^2) \tag{1.13}$$

with

$$\text{supp } \{\phi, \psi\} \subseteq \{ x \mid |x| \leq \rho \} \quad (\rho > 0 \text{ constant}) \tag{1.14}$$

and for $|\tilde{\lambda}| \leq 1$, where $\tilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2)$, $b_{ij}(\tilde{\lambda})$, $a_{0j}(\tilde{\lambda})$ and $F_0(\tilde{\lambda})$ are sufficiently smooth functions satisfying

$$b_{ij}(\tilde{\lambda}) = b_{ji}(\tilde{\lambda}) \quad (i, j = 1, 2) \tag{1.15}$$