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LONG-TIME ASYMPTOTIC BEHAVIOR OF LAX-FRIEDRICHS SCHEME

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Abstract In this paper we investigate the asymptotic stability of the discrete shocks of the Lax-Friedrichs scheme for hyperbolic systems of conservation laws. For single equations, we show that the discrete shocks of the Lax-Friedrichs scheme are asymptotically stable in the sense of l^2 and l^1. For the systems of conservation laws, if the summation of initial perturbations equals to zero, we show the l^2 stability and l^1 boundedness.

Key Words Lax-Friedrichs scheme; discrete travelling waves; asymptotic stability; hyperbolic conservation laws; energy method

Classifications 39A11, 35L65

1. Introduction

We consider hyperbolic systems of conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \tag{1.1}$$

Let $x_j = j\Delta x$, $t_n = n\Delta t$; Δx and Δt are respectively the space and time step sizes. Denote the approximation of $u(x_j, t_n)$ by u_j^n, the Lax-Friedrichs scheme (L-F scheme) is:

$$u_j^{n+1}-\frac{1}{2}(u_{j+1}^n+u_{j-1}^n)+\frac{\lambda}{2}(f(u_{j+1}^n)-f(u_{j-1}^n))=0 \tag{1.2}$$

where $\lambda = \Delta t/\Delta x$. Or, in general, we have the following scheme:

$$u_j^{n+1}-u_j^n+\frac{\lambda}{2}(f(u_{j+1}^n)-f(u_{j-1}^n))=\frac{\alpha}{2}(u_{j+1}^n-2u_j^n+u_{j-1}^n) \tag{1.3}$$

where $0 < \alpha \le 1$. When $\alpha = 1$, (1.3) is just (1.2).

The L-F scheme has been playing important roles both in the theory and numerical computations of hyperbolic conservation laws. In 1950's Oleinik [1] studied the existence for global solutions of single conservation laws using this scheme. In 1980's, Diperna [2] and Ding Xiaxi, Chen Guiqiang, Luo Peizhu [3] also used it to prove the existence of weak solutions with large amplitude for some 2×2 systems. The L-F scheme also played an important role in the development of the difference methods. It is a representative for monotone schemes. About the monotone schemes, there have been systematic theories [4]–[6].

On the asymptotic stability of the difference equations, Jennings first investigated the monotone schemes [7]. But the work is only restricted to the strictly monotone schemes-that is, if we denote the scheme as

$$u_j^{n+1} = G(u_{j-r}^n, u_{j-r+1}^n, \cdots, u_{j+t}^n) \tag{1.4}$$

then the first order derivatives of G about each of its arguments must be positive. Obviously, the L-F scheme does not satisfy this condition. Moreover, there are some mistakes among the stability theorem in [7]. As much as we know, Ralston pointed out the mistakes in [7] and made a correction in his unpublished work. Engquist and Osher, in their paper [8], quoted part of Ralston's results, but unfortunately, there are still some mistakes in this part of paper [8]. We shall show this at the end of this section.

For the L-F scheme approximating systems, Chern [9] has proved that the solutions of the scheme is asymptotically stable provided that the initial value is a constant state when $|x|$ is sufficiently large. Liu and Xin [10] have proved that, for scheme (1.3), if $0 < \alpha < 1$; the solutions of Riemann problem are single or multiple shocks; and if the summation of the initial perturbations equals to zero, then the scheme solutions are asymptotically stable. Besides, on other schemes, Majda and Ralston [11] have proved the existence of the travelling wave solutions for a class of schemes using the center manifold theorem. Smyrlis [12] has proved the asymptotic stability for the stationary discrete shocks of the Lax-Wendroff scheme. Szepessy [13] proved the asymptotic stability for a kind of implicit finite element schemes approximating systems. Yu [14] proved that under some conditions the Lax-Wendroff scheme can not have the travelling wave solutions.

The aim of the present paper is to study the asymptotic stability of the L-F scheme (1.2). We shall prove that its solution on the odd grid nodes and on the even grid nodes tends to two travelling waves respectively. We first consider scalar equations. Although on the grid with double space and time step size, the L-F scheme is strictly monotone, then following Ralston's unpublished work, one can get a kind of stability, in this paper we will show that the energy integration method gives a better result. The more important is, our method can be applied to systems of equations. We shall combine the method in [10] with the method we use for scalar equations to get the result on systems under the similar conditions in [10].

The organization of this paper is as follows: at the end of Section 1 we show that the asymptotically results in [7] and [8] can not be generally true. In Section 2 we shall prove that when the initial value is a small perturbation of a travelling wave the solution is l^2 asymptotically stable. In Section 3, we shall prove our l^1-stability for large perturbation. Finally, Section 4 contains the results for systems of equations.

Now, we discuss the stability results in [7] and [8].

Suppose (1.1) is a scalar equation. For convenience, we assume $f'' > 0$, u_r, u_l and