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THE SEMI-GLOBAL ISOMETRIC IMBEDDING IN R^3 OF TWO DIMENSIONAL RIEMANNIAN MANIFOLDS WITH GAUSSIAN CURVATURE CHANGING SIGN CLEANLY*

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Dedicated to the 70th birthday of Professor Zhou Yulin

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Abstract An abstract Riemannian metric $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ is given in $(u,v) \in [0,2\pi]\times[-\delta,\delta]$ where E, F, G are smooth functions of (u,v) and periodic in u with period 2π. Moreover $K|_{v=0} = 0$, $K_v|_{v=0} \neq 0$, where K is the Gaussian curvature. We imbed it semiglobally as the graph of a smooth surface $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ of R^3 in the neighborhood of $v = 0$.

In this paper we show that, if $[K_v\Gamma_{11}^2]_{v=0} < 0$ and three compatibility conditions are satisfied, then there exists such an isometric imbedding.

1. Introduction

Let

$$ds^2 = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2 \tag{1}$$

be a sufficiently smooth Riemannian metric in $(u,v) \in [0,2\pi]\times I_\delta$, where $I_\delta = [-\delta,\delta]$ and E, F, G are periodic functions of u with period 2π.

Consider the isometric imbedding problem in the neighborhood of $\Lambda = [0,2\pi]\times\{0\}$, i.e. realizing ds^2 in $[0,2\pi]\times I_{\delta_1}$ $(0<\delta_1<\delta)$ as the graph of a smooth surface $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ in R^3 such that $ds^2 = dx^2 + dy^2 + dz^2$. It is well known that the above problem was solved by [1], [2] for the cases of Gaussian curvature $K(u,v) > 0$ or $K(u,v) < 0$ respectively. And it was solved by [3] for the case $K(p) = 0$, $DK(p) \neq 0$ in $I_\eta \times \{0\}$ (η is small). In this paper we solve the isometric imbedding problem in the neighborhood of Λ with $K|_\Lambda = 0$, $K_v|_\Lambda \neq 0$ and K_v has different sign with Γ_{11}^2 on $v = 0$. In case K_v has the same sign with Γ_{11}^2, the semi-global imbedding problem is still open. The reason is, in the later case to solve z reduced to Tricomi mixed type equation, it is difficult to treat for periodic case, while for the former case, it reduced to a Buseman mixed type equation and easily to be solved.

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2. Necessary Conditions for Imbedding

It is well known that

$$K = -\frac{1}{2}(EG - F^2)^{-1}(E_{vv} - 2F_{uv} + G_{uu}) + \frac{1}{2}\Gamma^1_{22}E_v + \frac{1}{2}\Gamma^2_{12}G_u + \frac{1}{2}\Gamma^1_{11}(G_u - 2E_v) + \frac{1}{2}\Gamma^2_{11}G_v \tag{2}$$

where $\Gamma^i_{jk}(1 \le i, j, k \le 2)$ are the Christoffel symbols, i.e.

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}(GE_u - 2FF_u + FE_v)/(EG - F^2) \\
\Gamma^2_{11} &= \frac{1}{2}(2EF_u - FE_u - EE_v)/(EG - F^2) \\
\Gamma^1_{12} &= \frac{1}{2}(GE_v - FG_u)/(EG - F^2) \\
\Gamma^2_{12} &= \frac{1}{2}(FG_u - FE_v)/(EG - F^2) \\
\Gamma^1_{22} &= \frac{1}{2}(2GF_v - GG_u - FG_v)/(EG - F^2) \\
\Gamma^2_{22} &= \frac{1}{2}(FG_u - 2FF_v + EG_v)/(EG - F^2)
\end{aligned} \tag{3}$$

Let $z(u, v)$ be an arbitrary smooth function of u, v and let the metric g be

$$g = ds^2 - dz^2 = (E - z_u^2)du^2 + 2(F - z_u z_v)dudv + (G - z_v^2)dv^2$$

Assume that g is flat. It means the Gaussian curvature $K_g = 0$. The condition for $K_g = 0$ is equivalent to[3]

$$(z_{uu} - \Gamma^1_{11}z_u - \Gamma^2_{11}z_v)(z_{vv} - \Gamma^1_{22}z_u - \Gamma^2_{22}z_v) - (z_{uv} - \Gamma^1_{12}z_u - \Gamma^2_{12}z_v)^2 - [EG - F^2 - (Gz_u^2 - 2Fz_u z_v + Ez_v^2)]K = 0 \tag{4}$$

Theorem 1 *If there exist smooth isometric imbedding functions $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ in the small neighborhood of Λ and periodic with period 2π in u and $z = O(v^2)$, then we have*

$$[K_v\Gamma^2_{11}]_\Lambda < 0 \tag{5}$$

$$\int_0^{2\pi} [|\Gamma^2_{11}|(EG - F^2)^{1/2}/E]_{v=0}du = 2\pi \tag{6}$$

$$\int_0^{2\pi} E(u, 0)^{1/2} \exp\left\{\sqrt{-1}\int_0^u [|\Gamma^2_{11}|(EG - F^2)^{1/2}/E]_{v=0}du\right\} = 0 \tag{7}$$