*J. Partial Diff. Eqs.*, Vol.6, No.1 (1993), 80-96


# ON THE LOCAL REGULARITY OF SOLUTIONS FOR DOUBLE DEGENERATE NONLINEAR PARABOLIC EQUATIONS $(u^{q-1})_t=\text{div}\ (|\nabla u|^{p-2}\nabla u)$ WHEN $1 < p < 2, p \leq q$*

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Dedicated to the 70th birthday of Professor Zhou Yulin

(Received March 5, 1992; revised June 1, 1992)

**Abstract** In this paper, we establish interior Hölder estimates of solutions for double degenerate nonlinear parabolic equations $(u^{q-1})_t = \text{div}\ (|\nabla u|^{p-2}\nabla u)$ when $1 < p < 2, p \leq q$.

**Key Words** Hölder continuity; double degenerate; nonlinear parabolic equations
**Classifications** 35K55, 35K65

## 1. Introduction

In this paper, we are mainly concerned with local Hölder continuity of nonnegative weak solution for the following double degenerate parabolic equations

$$(u^{q-1})_t = \text{div}\ (|\nabla u|^{p-2}\nabla u) \quad \text{in } Q_T \tag{1.1}$$

where $1 < p < 2$, $p \leq q$, $Q_T = \Omega \times (0, T]$, $\Omega$ is an open set in $R^N (N \geq 1)$, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right)$.

For $p = 2$, (1.1) may be considered as the porous media equations $v_t = \Delta(v^{\frac{1}{q-1}})$ with $v = u^{q-1}$. Hölder continuity of solutions for porous media equations was proven in past, see [1], [2], [6], [7].

When $q = 2$, (1.1) is evolutionary $p$-Laplace equation, Hölder estimates for its weak solution and gradients of solutions have recently been obtained, see [3]-[6].

For double degenerate equations (1.1), the existence and uniqueness theorem and other properties of solutions have recently been investigated by some works, see [10]-[12]. When $1 < q \leq p$, $p > 2$, Hölder continuity of solutions of (1.1) has just been proven by the authors, see [8].

For a weak solution $u$ (supersolution, subsolution) of (1.1), we mean that $u \geq 0$, $u \in L^p(0, T; W^{1,p}(\Omega))$, $v, v_t \in L^2(Q_T)$, where $v = u^{q-1}$, and $u$ satisfies

$$\int_{t_1}^{t_2}\int_\Omega v_t\varphi dxdt = (\geq, \leq) \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dxdt \tag{1.2}$$

* The project supported by National Natural Science Foundation of China.

for $0 \le t_1 < t_2 \le T$, $\varphi \in L^2(Q_T) \cap L^p(0,T;W_0^{1,p}(\Omega)), \varphi \ge 0$.

Under appropriate conditions, one can prove the local boundedness of weak solution for (1.1). Throughout this paper we assume $0 \le u \le M$.

Our main result is the following.

**Theorem 1.1** *Assume that $u$ is a weak solution of (1.1) with $1 < p < 2$, $p \le q$, and $0 \le u \le M$. Then for any $\varepsilon \in (0,1)$, there exist constants $\beta, C > 0$ dependent only on $p$, $q$, $N$, $M$, $\varepsilon$, $0 < \beta < 1$, such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le C(|x_1 - x_2| + |t_1 - t_2|^{1/p})^\beta$$

*for all $(x_1,t)$, $(x_2,t_2) \in \Omega_\varepsilon \times (\varepsilon, T-\varepsilon)$, $\Omega_\varepsilon = \left\{x \in \Omega : |x| < \frac{1}{\varepsilon}, d(x,\partial\Omega) > \varepsilon\right\}$.*

## 2. Preliminary

In this section, we will give several Lemmas used later. Set

$$K_R(x_0) = \{x \in R^N : |x^i - x_0^i| \le R,\ 1 \le i \le N\}$$
$$Q(R,\rho;z_0) = K_R(x_0) \times (t_0 - \rho, t_0], z_0 = (x_0,t_0)$$

Assume $Q(R,\rho;z_0) \subset Q_T$.

**Lemma 2.1** *If $u$ is a supersolution of (1.1), then*

$$\sup_{t_0-\rho<t\le t_0} \int_{K_R(x_0)} \zeta^p\Big[\int_u^k s^{q-2}(k-s)^+ ds\Big]dx + \iint_{Q(R,\rho;z_0)} \zeta^p|\nabla(k-u)^+|^p dxdt$$
$$\le C\iint_{Q(R,\rho;z_0)} \Big\{|\nabla\zeta|^p(k-u)^{+p} + \zeta^{p-1}|\zeta_t|\Big[\int_u^k s^{q-2}(k-s)^+ ds\Big]\Big\}dxdt \quad (2.1)$$

*If $u$ is a subsolution of (1.1), then*

$$\sup_{t_0-\rho<t\le t_0} \int_{K_R(x_0)} \zeta^p\Big[\int_k^u s^{q-2}(s-k)^+ ds\Big]dx + \iint_{Q(R,\rho;z_0)} \zeta^p|\nabla(u-k)^+|^p dxdt$$
$$\le C\iint_{Q(R,\rho;z_0)} \Big\{|\nabla\zeta|^p(u-k)^{+p} + \zeta^{p-1}|\zeta_t|\Big[\int_k^u s^{q-2}(s-k)^+ ds\Big]\Big\}dxdt \quad (2.2)$$

*In (2.1) and (2.2), $k > 0$, constant $c$ depends only on $p,q$. $\zeta \ge 0$, $\zeta \in C^1(\bar{Q}(R,\rho;z_0))$, $\zeta|_{\partial_p Q(R,\rho;z_0)} = 0$.*

**Proof** In (1.2), by taking $\varphi = \zeta^p(k-u)^+$, we easily obtain (2.1). Similarly, (2.2) can be proven.

**Lemma 2.2** *For $1 < p < 2$, $q \ge p$, there exist $C_1$, $C_2$ dependent only on $q$, $p$ such that for $u \ge 0$*

$$C_1k^{q-2}(k-u)^{+2} \le \int_u^k s^{q-2}(k-s)^+ ds \le C_2k^{q-p}(k-u)^{+p} \quad (2.3)$$