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ON THE INTERACTION BETWEEN DISSIPATION AND SUPPLY AGENTS IN SOME NONLINEAR EVOLUTION PROBLEMS

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Abstract The paper deals with typical examples of semilinear evolution systems possessing solutions which behave at the interior of the considered domain unlike they do at its boundary. Sharp conditions are obtained which guarantee interior boundedness of a solution component despite the growth of corresponding data as well as ensure strict interior positivity of a solution component despite its decay at the boundary.

1. Introduction

Let Ω be a domain (unbounded, generally speaking) in $\boldsymbol{R}^n$, $x = (x_1, \cdots, x_n)$, $u = (u_1, \cdots, u_N)$, $F(u) = (F_1(u), \cdots, F_N(u))$, $\boldsymbol{R}_+ = (0, +\infty)$. Consider the boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = \sum_{\langle\alpha\rangle \le m} A_\alpha D^\alpha u(x,t) + F(u(x,t)), \quad (x,t) \in \Omega \times \boldsymbol{R}_+ \tag{1.1}$$

$$u(x,0) = \varphi(x), \quad x \in \bar{\Omega} \tag{1.2}$$

$$\sum_{\langle\alpha\rangle \le p} B_\alpha D^\alpha u(x,t) = \psi(x,t), (x,t) \in \partial\Omega \times \boldsymbol{R}_+ \tag{1.3}$$

Here $F \in [C^1(\boldsymbol{R}^n)]^N$, A_α are constant $(N \times N)$-matrices, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_i (i = 1, \cdots, n)$ are nonnegative integers, $\langle\alpha\rangle = \alpha_1 + \cdots + \alpha_n$, $D^\alpha u = \partial^{\langle\alpha\rangle} u / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, B_α are constant $(\mathcal{K} \times N)$-matrices for some $\mathcal{K}$, $\varphi : \overline{\Omega} \to \boldsymbol{R}^N$ and $\psi : (\partial\Omega \times \bar{\boldsymbol{R}}_+) \to \boldsymbol{R}^{\mathcal{K}}$ are prescribed functions. Suppose there exists a solution $u(x,t)$ of the problem (1.1)–(1.3) such that $u \in [L^\infty(\Omega \times (0, t^*))]^N$ for any finite $t^* > 0$.

We will be interested in the following two questions.

Question 1. Let (1.3) imply

$$\lim_{t \to +\infty} |u_N(x,t)| = +\infty \quad \forall x \in \partial\Omega$$

When does there exist a subset $\Omega^* \subset \Omega$ such that

$$\limsup_{t \to +\infty} |u_N(x,t)| < +\infty \quad \forall x \in \Omega^*$$

In other words, when does $u_N(x,t)$ remain bounded at certain interior points in spite of increasing data?

Question 2. Let (1.3) imply

$$\lim_{t\to+\infty} u_N(x,t) = 0 \quad \forall x \in \partial\Omega$$

When does there exist a subset $\Omega_* \subset \Omega$ such that

$$\liminf_{t\to+\infty} |u_N(x,t)| > 0 \quad \forall x \in \Omega_*?$$

In other words, when does $u_N(x,t)$ avoid extinction at certain interior points in spite of decaying data?

To the best of the present author's knowledge, topics related to Question 2 have not been explicitely discussed in the literature. On the contrary Question 1 involves the phenomenon of the *interior boundedness* which has been considered in a number of papers mainly for scalar second order parabolic equations. In [1] (see also [2]) this subject was studied for the classical linear heat equation with boundary data infinitely increasing as the time variable t tends to a *finite* value $T < +\infty$. In case $T = +\infty$ it was shown [3] that this phenomenon does not take place for the heat equation but may occur for a *nonautonomous* linear second order parabolic equation which contains lower order terms with unbounded variable coefficients having appropriate signs and rates of growth.

A phenomenon related to the interior boundedness is the so called *space localization*. By the definition it takes place if the support of the solution to a boundary value problem in $\Omega \times \bar{\boldsymbol{R}}_+$ with compactly supported initial data is contained in a cylinder $\Omega_0 \times \bar{\boldsymbol{R}}_+$ where diam $\Omega_0 < +\infty$. Clearly this property is nontrivial only for unbounded Ω. The space localization has been studied by a lot of authors mainly for scalar second order *quasi-linear* parabolic equations; see e.g. the pioneering paper [4], the book [5] and the survey article [6]. For the *semilinear* heat equation with absorption this phenomenon may occur only in the case of a *nonsmooth* lower order term.

With the aim to exclude the above mentioned possibilities we will restrict ourselves to the problems of the type (1.1)–(1.3) where $\psi(x,t)$ *is finite for any finite* t, matrices A_α have *constant* elements, and $F(u)$ *is smooth and independent of* x, t.

For typical special cases of the problem (1.1)–(1.3) we will give the conditions on data ensuring affirmative answers to the posed questions. With the aid of examples we will show the sharpness of the stated conditions.

The results presented here for model special cases may be extended to much more wide classes of problems. Such generalizations will be published elsewhere.

2. Interior dissipation *versus* boundary supply