J. Partial Diff. Eqs., Vol.6, No.2 (1993), 137–154

EXISTENCE AND ATTRACTIVITY OF STATIONARY SOLUTIONS OF A DEGENERATE DIFFUSION EQUATION*

Wu Yaping
(Institute of Applied Physics and Computational Mathematics, P.O.Box 8009, Beijing, China)

Ye Qixiao
(Dept. of Appl.Math., Beijing Institute of Technology, Beijing, China)

Dedicated to the 70th birthday of Professor Zhou Yulin

(Received March 27, 1992; revised June 29, 1992)

Abstract This paper is concerned with the existence and attractivity of the stationary solutions of a degenerate diffusion equation. By using monotone method and applying some special techniques, we extend the results in 1-dimension case to higher dimension cases.

Key Words Degenerate diffusion equation, existence, uniqueness, attractivity, radial solution

Classification 35B40, 35J25, 35J70, 35K65

1. Introduction

We want to investigate the following boundary value problem:

$$\begin{cases} \Delta(u^m) - a(|x|)u^p = 0, & x \in \boldsymbol{R}^n \\ u \geq 0, \quad u & \text{of compact support} \end{cases} \tag{1.1}$$

where $m > 1$, $n > 1$, $p \geq 1$ and $m > p$. As for a, the following properties will be assumed:

(A.1) $a(r) \in C^1([0, \infty))$ and $a'(r) > 0$ for any $r \in (0, \infty)$,

(A.2) there exists $\alpha > 0$ such that $(r - \alpha)a(r) \geq 0$ for any $r \in [0, \infty)$.

The initial value problem corresponding to (1.1) is the following Cauchy problem:

$$\begin{cases} u_t = \Delta(u^m) - a(|x|)u^p, & (t, x) \in (0, \infty) \times \boldsymbol{R}^n \\ u = u_0(x), & (t, x) \in \{0\} \times \boldsymbol{R}^n \end{cases} \tag{1.2}$$

We shall always consider $u_0 \geq 0$, and $u_0 \in L^\infty(R^n)$ with compact support.

Problems (1.1) and (1.2) were proposed as a model of mathematical population dynamics[1][2]. The existence of solutions both for (1.2) and for the Dirichlet initial boundary value problems was proved in [9]. The uniqueness of nontrivial solutions of (1.1) was proved in [4].

*The research supported by National Natural Science Foundation of China.

In this paper, we consider the existence of nontrivial radial solutions of (1.1). For this purpose we investigate the problem:

$$\begin{cases} y'' + \dfrac{n-1}{r}y' - a(r)y^{\beta} = 0, & r \in R_+ \\ y(0) = \eta, \quad y'(0) = 0 \end{cases} \tag{1.3}$$

where $\eta > 0$, $\beta = \dfrac{p}{m} \in (0,1)$, and a satisfies (A.1) and (A.2). Only classical nonnegative solutions of (1.3) will be considered.

For $n = 1$, problems (1.1)–(1.3) were investigated in [5], in which a complete description of the set of stationary solutions – as well as their attractivity properties – both for Cauchy problem (1.2) and for the Dirichlet or Neumann initial boundary value problems was given.

In this paper, we prove the existence and attractivity of solutions for (1.1) in higher dimension cases – as well as the existence of nontrivial radial solutions both for the Dirichlet and Neumann boundary value problems. We extends the results of [5] to higher dimension cases. It is worth mentioning that for $n = 1$ the attractivity holds in L^{∞}-norms[5][3], while for $n > 1$ the attractivity holds in L^q-norm for any $1 < q < \infty$. (see also [12]).

Due to the singularity appeared in (1.3), the methods used in [5] are not valid for $n > 1$. In this paper, the methods for the existence proof are sub-supper solutions method and a special maximum principle[11] as well as some special techniques. The attractivity results are also obtained by sub-supper solutions method (see also [12]). The very structure of sub-supper solutions makes some proofs easier than those in [5].

This paper is organized as follows: in Section 2 we give some basic lemmas, Section 3 is concerned with the initial value problem (1.3), in Section 4 the existence and attractivity of stationary solutions is obtained.

2. Basic Lemmas

Consider the initial value problem for radial equation:

$$\begin{cases} (r^{n-1}u_r(r))_r + r^{n-1}f(u(r),r) = 0, & r \in R_+ \\ u(0) = d > 0, \quad u_r(0) = 0 \end{cases} \tag{2.1}$$

Lemma 2.1[8] *Assume that $n \geq 1$ and that $f : (0,\infty) \times [0,\infty) \to R$ is C^1. For each $d > 0$ there exists a unique positive C^2 solution $u(r) = u(r,d)$ of (2.1). The function $u(r)$ is defined on a maximal interval $[0, r_d)$ and u is C^2 in the r variable and C^1 in the d variable.*

Lemma 2.2[11] **(Maximum Principle)** *If $u(x)$ satisfies the differential inequality:*

$$(L+h)[u] \equiv u'' + g(x)u' + h(x)u \geq 0 \tag{2.2}$$