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PERIODIC BOUNDARY PROBLEM AND CAUCHY PROBLEM FOR THE FLUID DYNAMIC EQUATION IN GEOPHYSICS*

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Abstract We study periodic boundary problem and Cauchy problem for the fluid dynamic equation in geophysics. The generalized and classical global solution of the mentioned problems are established. The method employed in this paper is Galerkin approximation and integral estimates.

Key Words periodic boundary problem; fluid dynamic equation; integral estimates; existence and uniqueness.

Classification 35Q20

1. Introduction

The fluid dynamic equation in geophysics is an equation of the form[1]

$$\psi_t - \Delta\psi_t + J(\psi, \Delta\psi) + A(\Delta\psi - \psi)_x + \psi_x = 0 \tag{1}$$

where $\psi(x, y, t)$ is the unknown function dependent on the time variable t and two space variables x and y, Δ is the two-dimensional Laplace operator, A is a constant and $J(\xi, \eta)$ is the determinent of the Jocobi derivative matrix, i.e., $J(\xi,\eta) = \xi_x\eta_y - \xi_y\eta_x$. This equation is of the pseudo-parabolic type and contains the strong nonlinear terms with derivatives of higher order, having some what nature of degenerality. The similar equation of this kind also appears in the study of dynamic theory of plasma, where the unknown function denotes the static electric potential[2].

In the present work, we are going to consider the equation of the generalized form

$$(u - \Delta u)_t + J(u, \Delta u) + A\Delta u_x + B\Delta u_y + f(u)_x + g(u)_y = h(u) \tag{2}$$

Firstly the periodic boundary problem in the domain $Q_T = \{(x, y) \in Q : 0 \le t \le T\}, Q = \{0 \le x, y \le 2D\}$ with the conditions

$$u(x, y, t) = u(x + 2D, y, t) = u(x, y + 2D, t) \tag{3}$$

for $(x, y) \in R^2$ and $0 \le t \le T$, and

$$u(x, y, 0) = \psi(x, y) \tag{4}$$

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is established by the method of Galerkin approximation, where A and B are constants; $f(\xi), g(\xi)$ and $h(\xi)$ are given functions for $\xi \in \mathbf{R}$ and $\psi(x,y)$ satisfies the periodic conditions (3).

The solution of the Cauchy problem for the equation (2) with the initial condition

$$u(x,y,0) = \psi(x,y) \tag{5}$$

in the domain $Q_T = \{(x,y) \in \mathbf{R}^2; 0 \le t \le T\}$, can be obtained by the limiting process while increasing the periodic of domain Q to infinity.

Let $\{y_k(x,y) | k = 1,2,\cdots\}$ be complete normalized orthogonal system of eigenfunctions corresponding to the eigenvalues $\{\lambda_k | k = 1,2,\cdots\}$ for the periodic boundary problem in Q for the equation $\Delta y = \lambda y$. Then the Galërkin approximate solution $u^N(x,y,t)$ for the problem (2), (3), (4) can be expressed as

$$u^N(x,y,t) = \sum_{k=1}^{N} \alpha_{N,k}(t) y_k(x,y) \tag{6}$$

where $\alpha_{N,k}(t)(k = 1,2,\cdots,N)$ are the coefficients to be determined and N is a natural number. According to the Galërkin method, the undetermined coefficients $\alpha_{N,s}(t)(s = 1,2,\cdots,N)$ satisfy the system of ordinary differential equations

$$\begin{aligned}(1-\lambda_s)\alpha_{N,s}(t) + (J(u^N,\Delta u^N), y_s) + (A\Delta u_x^N + B\Delta u_y^N \\ + f(u^N)_x + g(u^N)_y - h(u^N), y_s) = 0\end{aligned} \tag{7}$$

with the initial conditions

$$\alpha_{N,s}(0) = (\psi, y_s) \tag{8}$$

where $s = 1,2,\cdots,N$ and (u,v) denotes the integral $\iint_Q u(x,y)v(x,y)dxdy$ as usual.

We also adopt the similar notations and abbreviations as used in [3-6].

2. Estimates for Approximate Solutions

Lemma 1 *Suppose that $f'(\xi), g'(\xi), h(\xi) \in C^0(\mathbf{R})$ and*

$$\xi h(\xi) \le A_0 \xi^2 \tag{9}$$

for $\xi \in \mathbf{R}$ and $\psi(x,y) \in H^1(Q)$ satisfying the periodic condition (3), where A_0 is a constant. Then there is the estimate for the approximate solution $u^N(x,t)$ as:

$$\sup_{0\le t\le T} \|u(\cdot,\cdot,t)\|_{H^1(Q)} \le K_1 \|\varphi\|_{H^1(Q)} \tag{10}$$

where K_1 is a constant independent of N and Q.

Proof Multiplying (7) by $2\alpha_{N,s}(t)$ and summing up the products for $s = 1,2,\cdots,N$, we get

$$\begin{aligned}2(u_t^N - \Delta u_t^N, u^N) + 2(J(u^N,\Delta u^N), u^N) + 2(A\Delta u_x^N + B\Delta u_y^N \\ + f(u^N)_x + g(u^N)_y - h(u^N), u^N) = 0\end{aligned}$$