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$C^{1,\alpha}$ REGULARITY OF VISCOSITY SOLUTIONS OF FULLY NONLINEAR ELLIPTIC PDE UNDER NATURAL STRUCTURE CONDITIONS[1]

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Dedicated to the 70th birthday of Professor Zhou Yulin

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Abstract In this paper we are concerned with fully nonlinear elliptic equation $F(x,u,Du,D^2u)=0$. We establish the interior Lipschitz continuity and $C^{1,\alpha}$ regularity of viscosity solutions under natural structure conditions without differentiating the equation as usual, especially we give a new analytic Harnack inequality approach to $C^{1,\alpha}$ estimate for viscosity solutions instead of the geometric approach given by L. Caffarelli & L. Wang and improve their results. Our structure conditions are rather mild.

1. Introduction

In this paper we are concerned with the Dirichlet problem for fully nonlinear elliptic equation

$$F(x,u,Du,D^2u)=0 \quad \text{in } \Omega \tag{1.1}$$

$$u=g \quad \text{on } \partial\Omega \tag{1.2}$$

where Ω is an open bounded domain in R^N. $F(x,r,p,X)$ is a function on $\Gamma=\Omega\times R\times R^N\times M^N$, where M^N denotes the space of $N\times N$ symmetric matrices equipped with usual order. Du and D^2u are, respectively, the gradient and the Hessian of $u(x)$. We always suppose F is continuous on Γ.

The purpose of this paper is to establish Lipschitz continuity and $C^{1,\alpha}$ interior regularity of viscosity solutions of (1.1) under the natural structure conditions and without the concave condition on F with respect to X.

In [5], Dong and Bian obtained $C^{1,\alpha}$ regularity of viscosity solutions of (1.1) but some structure conditions are not satisfactory, especially the condition

$$|F_p(x,r,p,X)|\le \mu_1(|r|,|p|)+\mu_0\|X\|$$

where μ_0 is small enough. Using geometric method Caffarelli & Wang [3] get $C^{1,\alpha}$ regularity of viscosity solutions of $F(Du,D^2u)=0$. They assume that the solutions

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are Lipschitz continuous and $F(p, X)$ is Lipschitz continuous with respect to p. In this paper, following their some idea and extending the technique in [6] we give a new analytic Harnack inequality approach to $C^{1,\alpha}$ regularity of viscosity solutions without differentiating the equation (1.1) as usual and improve the result in [3] (see (1.12)). Our structure conditions for $C^{1,\alpha}$ regularity are rather mild and natural.

First we recall the definition of viscosity solutions.

Definition 1.1 *Let u be an upper semi-continuous function (resp. lower semi-continuous) on Ω. u is said to be a viscosity subsolution (resp. super-solution) of (1.1) if for all $\varphi \in C^2(\Omega)$ the following inequality*

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \le 0$$

$$(\text{resp. } F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \ge 0)$$

holds at each local maximum (resp. minimum) point $x_0 \in \Omega$ of $u - \varphi$. Then $u \in C(\Omega)$ is said to be a viscosity solution of (1.1) if u is a viscosity subsolution and a viscosity supersolution.

Now we describe our main results.

As a basis of our discussion we first establish the existence results about viscosity solutions. Ishii and Lions [6] have obtained very beautiful comparison results for strictly elliptic equations. Then by using Perron's method they obtain the existence of viscosity solution of the problem (1.1), (1.2) if assuming that there exist viscosity sub-and super-solutions satisfying (1.2). Obviously it is not easy to check this assumption. Using the barrier argument we give a concrete existence result for uniformly elliptic equations.

Consider the structure conditions:

$$\lambda(|r|,|p|)\mathrm{Tr}(Y) \le F(x,r,p,X) - F(x,r,p,X+Y) \le \Lambda(|r|,|p|)\mathrm{Tr}(Y)$$

$$\lambda(|r|,|p|) > 0 \quad \Lambda(|r|,|p|)/\lambda(|r|,|p|) \le \gamma_0(|r|), Y \in M^N, Y \ge 0 \tag{1.3}$$

$$\begin{aligned}&|F(x,r,p,X) - F(y,r,p,X)| \\ &\quad \le \gamma_1(|r|)\lambda(|r|,|p|)\{1 + \omega(|x-y|)|x-y|^\tau |p|^{2+\tau} + \mu(|x-y|)\|X\|\}\end{aligned} \tag{1.4}$$

$$\begin{aligned}&|F(x,r,p,X) - F(y,r,p,X)| \\ &\quad \le \gamma_2(|r|,|p|)\lambda(|r|,|p|)\omega(|x-y|^\theta(1+\|X\|))\end{aligned} \tag{1.5}$$

for all $(x,r,p,X) \in \Gamma, y \in \Omega, \tau \in [0,1], \theta \in \left(\frac{1}{2}, 1\right]$, where $\gamma_i(\cdot)(i = 0,1), \gamma_2(\cdot,\cdot), \omega(\cdot)$ and $\mu(\cdot)$ are nondecreasing with respect to their variables on $[0,\infty), \omega(\sigma), \mu(\sigma) \to 0$ as $\sigma \to 0, \mu(\sigma)/\sigma \ge 1$ in $(0,\infty)$ and $\int_{0+} \frac{\mu(\sigma)}{\sigma} d\sigma < \infty$.

$$\exists \theta_M > 0 \quad s.t. \quad F(x,r,p,X) - F(x,s,p,X) \ge \theta_M(r-s) \tag{1.6}$$

for all $x, y \in \Omega, -M \le s \le r \le M, (p,X) \in R^N \times M^N$. If $\theta_M = 0$ for all $M > 0$ we denote it by $(1.6)_w$.

$$|F(x,r,p,0)| \le \lambda(|r|,|p|)\gamma_3(|r|)(1+|p|^2) \tag{1.7}$$