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THE GEOMETRIC MEASURE THEORETICAL CHARACTERIZATION OF VISCOSITY SOLUTIONS TO PARABOLIC MONGE-AMPÈRE TYPE EQUATION*

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Abstract By an involved approach a geometric measure theory is established for parabolic Monge-Ampère operator acting on convex-monotone functions, the theory bears complete analogy with Aleksandrov's classical ones for elliptic Monge-Ampère operator acting on convex functions. The identity of solutions in weak and viscosity sense to parabolic Monge-Ampère equation is proved. A general result on existence and uniqueness of weak solution to BVP for this equation is also obtained.

Key Words geometric measure theory; parabolic Monge-Ampère operator; weak (or generalized) and viscosity solution.
Classifications 35K20; 35K55; 35Q99.

0. Introduction

In this paper we give a full exposition of the results announced in [1] and [2], concerning the parabolic type Monge-Ampère operator acting on continuous functions $u(x,t)$, which are convex in $x \in \Omega \subset R^n$ and non-increasing in $t \in (0,T]$, with Ω being a convex domain. These objects were introduced by N.V.Krylov in [3], and since then have become basic tools in the study of parabolic equations.

Firstly we shall show that it is possible to establish a geometric measure theory in the present time dependent context bearing complete analogy with the Aleksandrov's classical ones for elliptic Monge-Ampère operator acting on the usual convex functions in Section 1. The starting point is that, as it was hinted in Tso's work [4], the proper substitute needed here, for the basic notion of normal image in Aleksandrov's construction, happens to be the set valued mapping, which will be called the Legendre transformation $\mathcal{L}$ generated by $u(x,t)$:

$$\mathcal{L} : (y,\tau) \in Q := \Omega \times (0,T] \to (p,h) \in R^n \times R \tag{0.1}$$

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with

$$p \in \nabla u(y,\tau) \quad h = p \cdot y - u(y,\tau) \tag{0.2}$$

where $\nabla u(y,\tau)$ denotes the sub-gradient of the convex function $u(\cdot,\tau)$ at y. Geometrically, (0.2) means that $z = p \cdot x - h$ is one of the supporting planes to the graph of $u(\cdot,\tau)$ passing through the point $(y, u(y,\tau))$.

Secondly we shall show that, for strictly positive functions $f(x,t) \in C(Q)$, a function $u(x,t) \in C(Q)$, convex in x and non-increasing in t, is a solution of the parabolic Monge-Ampère equation

$$-u_t(x,t)\det[D_x^2 u(x,t)] = f(x,t) \quad \text{in } Q \tag{0.3}$$

understood in the above mentioned Aleksandrov's measure theoretical like weak sense, if and only if it is a solution in viscosity sense in Section 2. A result of this kind for elliptic Monge-Ampère equation was proved by Caffarelli in [5].

By the way we shall discuss the first BVP

$$\begin{cases} -u_t(x,t)\det[D_x^2 u(x,t)] = f(x,t) & \text{in } Q \\ u(x,t) = \phi(x,t) & \text{on } \partial_p Q \end{cases} \tag{0.4}$$

Suppose Ω is a C^2 bounded convex domain, $f \in C(\bar{Q})$, $f(x,t) > 0\ \forall (x,t) \in \bar{Q}$, $\phi(x,t)$ is a continuous function defined in a neighborhood of $\bar{Q}$ and is convex in x and non-increasing in t. Then based on [1] we prove that there exists a unique weak solution to the first BVP (0.4).

The precise statements of the above mentioned results are the following two theorems.

Theorem A i) *For $u(x,t) \in C(Q)$, convex in x and non-increasing in t, consider the Legendre transformation $\mathcal{L}$ generated by $u(x,t)$ as defined by* (0.1) *and* (0.2). *Despite that $\mathcal{L}$ is a set-valued mapping in general, the set function by associating any Borel set $S \subset Q$ to the $(n+1)$-dimensional Lebesgue measure of $\mathcal{L}(S)$ is a Radon measure, ω_u, on Q.*

ii) *As it has been proved by N.V.Krylov that, for the function $u(x,t)$ as described in* i) *it holds, for almost all $(x,t) \in Q$, that*

$$u(x+y, t+\tau) = u(x,t) + u_t(x,t)\tau + \nabla_x u(x,t)\cdot y + \frac{1}{2}D_x^2 u(x,t)y\cdot y + o(|y|^2 + |\tau|) \quad as\ |y|^2 + \tau \to 0,$$

where $D_x^2 u(x,t)$ actually denotes the Radon-Nikodym derivatives of the measures $u_{x_i x_j}(\cdot,t)$ with respect to Lebesgue measure. Moreover the Radon-Nikodym derivative of ω_u is equal to

$$-u_t(x,t)\det[D_x^2 u(x,t)], \quad a.e. \quad on\ Q$$

iii) *Let $u_j(x,t) \in C(Q)$ be convex in x and non-increasing in t, ω_{u_j} be the measure associated with it as defined in* i). *For the function $u(x,t)$ and the measure ω_u as stated*