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ON THE LOCAL BEHAVIOUR OF PARABOLIC Q-MINIMA*

Zhou Shulin
(Department of Mathematics, Peking University, Beijing, 100871, China)
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Abstract In this paper, we have generalized the notion of parabolic Q-minima in [1] to provide a unifying approach to obtain some regularity results for certain nonlinear degenerate parabolic equations.

Key Words parabolic Q-minima; boundedness; Hölder continuity.
Classifications 35B45, 35K55, 35K65.

1. Introduction

M.Giaquinta and E.Giusti first introduced the notion of Q-minima to study elliptic problems. W.Wieser generalized this notion to the parabolic case, but his parabolic Q-minima are too narrow to contain the weak solutions of a large class of nonlinear degenerate parabolic equations, such as

$$u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u) \quad (p > 1)$$

Therefore we generalize parabolic Q-minima further to cover the weak solutions of the above equation and its general form.

We will consider our problems in the following Banach spaces:

$$V_{2,p}(\Omega_T) \equiv L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$$

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with the norm

$$\|u\|_{V_{2,p}(\Omega_T)} \equiv \operatorname*{ess\,sup}_{0\le t\le T} \|u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla_x u\|_{L^p(\Omega_T)}$$

and

$$\|u\|_{\overset{\circ}{V}_{2,p}(\Omega_T)} \equiv \|u\|_{V_{2,p}(\Omega_T)}$$

where Ω is a bounded open domain in $\boldsymbol{R}^N$, Ω_T the cylinder $\Omega \times (0,T), T \in (0,+\infty)$ and $W^{1,p}(\Omega)(\overset{\circ}{W}{}^{1,p}(\Omega))$ is the usual Sobolev space.

Definition *A n-dimentional vector function $v \in L^p_{\rm loc}(0,T;W^{1,p}_{\rm loc}(\Omega)) \cap L^\gamma_{\rm loc}(\Omega_T)$ is called a parabolic Q-minimum if there exist a constant $Q \ge 1$, and a Carathéodory*

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function $F = F(x,t,v,q) : \Omega \times (0,T) \times R^n \times R^{nN} \to R^n$, *such that for every* $\phi \in C_0^\infty(\Omega_T)$, *the following inequality*

$$-\int_K u\frac{\partial\phi}{\partial t}dz + E(u;K) \le QE(u-\phi;K) \tag{1.1}$$

holds, here $K = \mathrm{spt}\phi, z = (x,t), dz = dxdt$, *and*

$$E(v;K) = \int_K F(z,v,\nabla v)dz \tag{1.2}$$

and F *satisfies the growth condition:*

$$\lambda_1|q|^p - b|v|^\gamma - g(x,t) \le F(z,v,q) \le \lambda_2|q|^p + b|v|^\gamma + g(x,t) \tag{1.3}$$

where $\lambda_1, \lambda_2 > 0, b \ge 0, p > 1, \gamma > 0$, *and* $g(x,t)$ *is a nonnegative integrable function. If* (1.1) *holds only for* ϕ *of a certain subset* $B \subset C_0^\infty(\Omega_T)$, u *is called a B-restricted parabolic Q-minimum.*

Remark 1.1 Set $\tilde F(x,t,v,q) = F(x,t,-v,-q)$, then $\tilde F$ satisfies (1.3). If u is a parabolic Q-minimum, let $\phi = -\psi \in C_0^\infty(\Omega_T)$, then there is fulfilled

$$-\int_K (-u)\frac{\partial\psi}{\partial t}dz + \tilde E(-u,K) \le Q\tilde E(-u-\psi,K)$$

where $K = \mathrm{spt}\phi = \mathrm{spt}\psi$, and

$$\tilde E(v,K) = \int_K \tilde F(z,v,\nabla v)dz$$

Therefore $-u$ is a parabolic Q-minimum.

We are mainly concerned with the problems of the scalar case for $n = 1$. For the simplicity of exposition, we will assume that F satisfies

$$\lambda_1|q|^p - C_0 \le F(z,v,q) \le \lambda_2|q|^p + C_0 \tag{1.4}$$

where $C_0 \ge 0, \lambda_1, \lambda_2 > 0$. The other cases when $g(x,t)$ and γ satisfy adequate conditions may be treated by the same arguments. (see [1], [2])

Our results are stated as follows:

Theorem 1 *Suppose* $p > \dfrac{2N}{N+2}$, *then any parabolic Q-minimum* $u \in V_{2,p,\mathrm{loc}}(\Omega_T)$ *is locally bounded in* Ω_T.

Theorem 2 *Suppose* $p \ge 2$, *then any parabolic Q-minimum* $u \in V_{2,p,\mathrm{loc}}(\Omega_T)$ *is locally Hölder continuous in* Ω_T *and for every compact set* $\mathcal{X} \subset\subset \Omega_T$, *there exist a constant* γ *depending only upon the data and* $\mathrm{dist}\{\mathcal{X}, \partial^*\Omega_T\}$, *and a constant* $\alpha \in (0,1)$ *depending only upon the data, such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{p}})$$

for every pair of points $(x_1,t_1),(x_2,t_2)$. *Here* $\partial^*\Omega_T = \partial\Omega_T \setminus \bar\Omega(T)$.