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ON THE SECOND–ORDER EQUATIONS OF MIXED TYPE OF FIRST KIND

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Abstract In this paper the generalized Tricomi problem for the second-order equation of mixed type of first kind is considered. The uniqueness of solutions is proved under very weak conditions on the coefficients of equation and the boundary curve of domain. The existence of H^1 strong solutions is proved for the Tricomi problem.

Key words Equation of mixed type; generalized Tricomi problem; strong solution.

Classification 35M05.

1. Introduction

Consider the equation of mixed type

$$Lu \equiv k(x,y)u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = f \tag{1}$$

where the function $k(x,y)$ satisfies the conditions: $yk > 0$ when $y \neq 0$ and $k(x,0) = 0$.

Let the outer boundary of $D^+(= D \cap \{y > 0\})$ be an arbitrary piecewise smooth curve Γ_0, which is connected with the degenerate line $y = 0$ at points A and B. Let the outer boundary of $D^-(= D \cap \{y < 0\})$ be two families of characteristic curves Γ_+ and Γ_-, issuing from the corresponding points A and B respectively and defined by the equations $dx + \sqrt{-k}dy = 0$ and $dx - \sqrt{-k}dy = 0$ respectively. Let Γ'_+ be an arbitrary piecewise smooth curve, issuing from the intersection point A of Γ_+ and Γ_0, and lying inside the characteristic triangle, but the slope of the curve Γ'_+ is not less than the slope of the characteristic line of the family Γ_+ at the corresponding points, and Γ'_+ is not tangent to the family Γ_-.

For Eq. (1) we consider the generalized Tricomi problem T' (or Tricomi problem T):

$$u = 0 \quad \text{on } \Gamma_0 \cup \Gamma'_+ \ (\text{or } \Gamma_0 \cup \Gamma_+) \tag{2}$$

Many authors have studied the problem (1) (2) (for example [1]–[13]), but most of them discussed this problem under many restrictions on the coefficients of equations and the curve Γ_0 in order to prove the uniqueness of solutions, whether for the simple equations or for the more complicated equations. In [12] [13] for the general equations containing an eigenvalue parameter λ, the existence and uniqueness of solutions to the generalized Tricomi problem had been proved, except a countable set of λ, but it had not reached the conclusion on λ for which the problem will be solvable. Hence for concrete equations the problem is still open.

The purpose of this paper is to weaken the restrictions on coefficients as far as possible and to give the more pleased conclusion. About the question on existence of solutions, in general, in literature only the weak solution was obtained, but in this paper the strong solution is gotten.

2. Uniqueness of Solutions

To prove the uniqueness of solutions for the problem (1) (2) it means we have to prove that the generalized Tricomi problem for the homogeneous equation

$$Lu \equiv k(x,y)u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = 0 \tag{3}$$

has zero solution only. For this, we have to seek for such solution u, which is continuous in the closed domain $\bar{D}$, and has first order continuous partial derivatives everywhere except the isolated points on boundaries of domain. In the neighborhood of the isolated points the order of its singularities is less than 1/2.

Now we firstly assume that

$$\beta = \beta(y) \in C^1(D), \quad \gamma = \gamma(y) \in C(D) \tag{4}$$

Make transformation

$$w(x,y) = u(x,y) \exp \int \mu(y)dy \tag{5}$$

where the function $\mu(y)$ satisfies a Riccati equation

$$\mu' = \mu^2 - \beta(y)\mu + \gamma(y) \tag{6}$$

If in the region of values of y in domain D (i.e., $y \in [-y_1, y_2]$, $-y_1 \le \min_{y \in \overline{D^-}} y < 0$, $y_2 \ge \max_{y \in \overline{D^+}} y > 0$, y_1 and y_2 are arbitrary positive real numbers.) there exists a C^1 solution for Eq. (6), then by the transformation (5) we can transform Eq. (3) into the following:

$$\tilde{L}w \equiv k(x,y)w_{xx} + w_{yy} + \alpha w_x + \lambda(y)w_y = 0 \tag{7}$$

where

$$\lambda(y) = \beta(y) - 2\mu(y) \in C^1(D) \tag{8}$$

Under the transformation(5) the boundary condition (2) becomes

$$w|_{\Gamma \cup \Gamma'_+ \text{ (or } \Gamma_+)} = 0 \tag{9}$$