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AN INVARIANT GROUP OF MKdV EQUATION*

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Abstract In this paper, we present an invariant group of the MKdV equation $q_t = q_{xxx} - 6q^2q_x$. By using this invariant group, we can obtain some new solutions from a known solution by quadrature.
Key Words Invariant group; MKdV equation; Miura transformation.
Classification 35Q.

It is meaningful but difficult to find the invariant groups of a differential equation. For the Nonlinear Evolution Equations, only a few invariant groups we have known. For example, there are four invariant groups of KdV equation: x-translation, t-translation, Galilean transformation and scalar transformation [1]. In this paper, we present an invariant group of MKdV equation.

In the following, $\int f dx$ (or $\int f dt$) means and arbitrary primitive function of f and it is taken definitely.

1. MKdV equation

$$q_t = q_{xxx} - 6q^2 q_x \tag{1.1}$$

is related to the KdV equation

$$u_t = u_{xxx} + 6uu_x \tag{1.2}$$

Between (1.1) and (1.2), there are the Miura transformations:

$$\mu_1 : u = -q_x - q^2$$

and

$$\mu_2 : u = q_x - q^2$$

Lemma 1.1 *If q is a solution of (1.1), then*

$$c \equiv \Big(\int q dx\Big)_t - (q_{xx} - 2q^3)$$

$$h \equiv -\Big(\int e^{-2\int q dx} dx\Big)_t - 2(q_x + q^2) e^{-2\int q dx} - 2c \int e^{-2\int q dx} dx$$

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are the functions of t only.

Proof It is not difficult to check

$$c_x = 0 = h_x$$

when q satisfies (1.1).

Theorem 1.1 *If q is a solution of* (1.1), *then*

$$\bar{q} = q + r \tag{1.3}$$

is a solution of (1.1) *as well, where*

$$r = e^{-2\int q dx} \Big/ \Big(\int e^{-2\int q dx} dx + \varepsilon(t) \Big) \tag{1.4}$$

$$\varepsilon(t) = e^{-2\int c(t) dt} \Big(\int h(t) e^{2\int c(t) dt} dt + \alpha \Big) \tag{1.5}$$

α is an arbitrary constant (i.e., $\varepsilon' = -2c\varepsilon + h$).

Proof We can check

$$\bar{q}_t = \bar{q}_{xxx} - 6\bar{q}\bar{q}_x \tag{1.6}$$

Substitute (1.3)–(1.5) into (1.6). Since

$$\bar{q}_{xxx} - 6\bar{q}^2\bar{q}_x = q_{xxx} - 6q^2 q_x - 2r(q_{xx} - 2q^3) + 2r^2(q_x + q^2)$$

and

$$q_t = q_{xxx} - 6q^2 q_x$$

$$\bar{q}_t = q_t + r_t$$

we only need to prove

$$r_t = -2r(q_{xx} - 2q^3) + 2r^2(q_x + q^2) \tag{1.7}$$

Substituting

$$r_t = -2r\Big(\int q dx \Big)_t - r^2 e^{2\int q dx} \Big(\Big(\int e^{-2\int q dx} dx \Big)_t + \varepsilon'(t) \Big)$$

into (1.7), we have

$$2c(t)e^{-2\int q dx} - 2rc(t) \int e^{-2\int q dx} dx - rh(t) + r\varepsilon'(t) = 0 \tag{1.8}$$

Substitute (1.4) into (1.8), (1.8) is reduced to

$$\varepsilon'(t) + 2c(t)\varepsilon(t) - h(t) = 0$$

The theorem is proved.