J. Partial Diff. Eqs., *Vol.6*, *No.4* (1993), 289-296
©*International Academic Publishers*

ON THE ANORMAL UNIQUENESS FOR A CLASS OF FIRST ORDER COUPLED ELLIPTIC SYSTEM

Du Xinhua
(Sichuan Normal University, Chengdu)
(Received Apr. 3, 1990)

Abstract In this paper, we studied and completely solved the problem on the anormal uniqueness of the equation system

$$\begin{cases} u_x + ixu_y = -ix^2\phi v_y \\ v_x + ixv_y = -ix^2\phi u_y \end{cases}$$

The so-called anormal uniqueness means as follows: suppose all order derivatives of u vanish on ∂D, one can draw the conclusions of $u \equiv 0(D)$, $v \equiv \text{const(D)}$.
Key Words Coupled elliptic system; Anormal uniqueness.
Classification 35J.

1. Introduction

In the solvability of the partial differential operators, the solvability of first order operators

$$P = \sum_{j=1}^{n} a^j(x)\Big(\frac{\partial}{\partial x^j}\Big)$$

was studied deeply. If $a^j(x)$ make complex values, note that $P = P_1 + iP_2$, where P_1, P_2 are linear independent, may induce some theory which depends analytic function. In case $n > 2$, it is very interesting that some very good operators P may lead to the fact that $Pw = f$ have not smooth solutions even distribution solutions for any $f \in C^\infty$ and in any open set. The famous example is Lewy's Operator[1]:

$$P = \frac{1}{2}\Big(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\Big) + i(x^1 + ix^2)\frac{\partial}{\partial x^3} \tag{1}$$

possesses this property. If we suitably choose f, then the simpler Grushin's operator:

$$P = \frac{\partial}{\partial x} + ix\frac{\partial}{\partial y} \tag{2}$$

will lead to the fact that $Pw = f$ havn't solution as well.

In fact[2], note that $D_n (n = 1, 2, \cdots)$ are arbitrary closed and mutually disjointed disc sequence in the right semi-plane $x > 0$ of (x, y)-plane, the centre of D_n are $(x_n, 0)$,

$x_n > 0$ and $x_n \to 0$ $(n \to \infty)$. Suppose $f(x,y) \in C^\infty$ is an arbitrary function that possesses compact support, and is an even function respecting x, which equals to zero outside D_n and $x \geq 0$, so that

$$\iint_{D_n} f dxdy \neq 0, \quad n = 1, 2, \cdots$$

It is easy to construct such function f. Thus one will be able to induce the following result: For the above-mentioned function f, the equation

$$\frac{\partial w}{\partial x} + ix\frac{\partial w}{\partial y} = f(x,y)$$

has not any solution in any neighbourhood of origin $(0,0)$.

As Lewy pointed out, if $Pw = f$ havn't any solution in any open set, then $(P-f)w = 0$ have uniqueness solution $w \equiv 0$. Therefore, Lewy rose the following problem: The first order homogeneous equation

$$Pw = \sum_{j=1}^{n} a^j \frac{\partial w}{\partial x^j} = 0, \quad \sum |a^j| \neq 0$$

whether always have the local non-trivial solution, or exist such operator that possesses uniqueness local solution $w \equiv C$ (const.)? Hörmander[3] pointed out: If P_1, P_2, and $[P_1, P_2]$ are linear independent, then there is a following problem: Can we find two solutions of $Pw = 0$ so that possess linear independent gradient, or can we find the solution w so that grad $w \neq 0$? L. Nirenberg[2] studied the previous problem, and gave the result: If

$$P = \frac{\partial}{\partial x} + ix\rho(x,y)\frac{\partial}{\partial y}, \quad \rho(x,y) \equiv 1 + x\phi(x,y)$$

then we are able to construct suitable ϕ so that any solution of $Pw = 0$ must be constant.

In L.Nirenberg's work, there is a strangle problem: On the mutually disjointed disc sequence $D_j^{m,n}$ (see [2]. In the sequel, we note one of $D_j^{m,n}$ as D), give a coupled elliptic system:

$$\begin{cases} u_x + ixu_y = -ix^2\phi v_y \\ v_x + ixv_y = -ix^2\phi u_y \end{cases} \tag{1}$$

where $\phi(x,y) \in C^\infty$, and $\phi \equiv 0$ outside D. Suppose all order derivatives of u vanish on ∂D, prove

$$u|_D \equiv 0, \quad v|_D \equiv c(\text{const.}) \tag{2}$$

Theorem 1 ("uniqueness") *If the solution $(u, v)$of the coupled elliptic system* (1) *satisfies the condition*

$$\partial^j_{x^\alpha y^\beta} u|_{\partial D} = 0, \quad \alpha + \beta = j, \quad j = 0, 1, 2, \cdots \tag{3}$$

then (u, v) possesses the property (2).