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L^p-ESTIMATES FOR THE STRONG SOLUTIONS OF ELLIPTIC EQUATIONS OF NONDIVERGENT TYPE*

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Abstract We investigate the second derivatives L^p-estimates for the strong solutions of second order linear elliptic equations in nondivergence form $Lu = f$ in the case in which the leading coefficients of L are not continuous. The L^p-estimates for small p are obtained if L is uniformly elliptic. Furthermore, if the leading coefficients of L belong to $W^{1,n}$, then we get the second derivatives L^p-estimates for large p. The existence of the strong solutions of the homogeneous Dirichlet problem is also considered.
Key Words Second derivatives L^p-estimates; strong solutions; discontinuous leading coefficients; perturbation technique; elliptic equations.
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1. Introduction

Let Ω be a smooth, bounded domain in R^n and $a(x) = (a_{ij}(x))$ be a symmetric measurable $n \times n$ matrix valued function in Ω which satisfies

$$\Lambda^{-1} I \le a(x) \le \Lambda I \quad \text{a.e.} \quad x \in \Omega \tag{1.1}$$

for some positive constant Λ. Set

$$L = L_a = \sum a_{ij}(x) D_{ij}$$

and let $u \in W^{2,n}$ be a strong solution of the following equation

$$Lu = f(x) \quad \text{in} \quad \Omega \tag{1.2}$$

where f is a function defined in Ω.

Let us consider *a priori* estimates of second derivatives in L^p and the homogeneous Dirichlet problem of Equation (1.2). It is well known[1] that if $a(x)$ is continuous and $f \in L^p, 1 < p < \infty$, then the strong solutions of (1.2) are of class $W^{2,p}$. The situation is very different if $a(x)$ is not continuous. See, for example, [2, 3 and 4]. F.H.Lin[5] has proved *a priori* estimate of second derivatives in L^p for some small $p > 0$. F. Mandras

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and G. Porru[7] proved that if $a(x) \in W^{1,n}$ and $f \in L^p$, then the solutions of (1.2) are in $W^{2,p}$ provided p belongs to a suitable neighbourhood of 2. B.J.Bian[8] has studied the Hölder gradient estimates for the strong solutions of (1.2).

In this paper we investigate *a priori* estimates of second derivatives in L^p by using the small perturbation technique. This technique is from [9] where L.A. Caffarelli studied the regularity theory of viscosity solutions of fully nonlinear equations.

Let u be a strong solution of Equation (1.2) with $u = 0$ on $\partial\Omega$ and (1.1) hold.

Theorem 1.1 *There exist positive constants $p = p(n, \Lambda)$, $C = C(n, \Lambda, \Omega)$ such that*

$$\|D^2 u\|_{p,\Omega} \leq C\|f\|_{n,\Omega} \tag{1.3}$$

The above theorem has been proved in [5]. Our method is different from that used in [5]. The main aim of this paper is to prove the following.

Theorem 1.2 *Assume that $a(x)$ belongs to $W^{1,n}$. Then*

(a) *Let $f \in L^p$ for some $p > n$, then there exists a constant $C = C(n, \Lambda, p, \Omega, a(x))$ such that*

$$\|D^2 u\|_{p,\Omega} \leq C\|f\|_{p,\Omega} \tag{1.4}$$

(b) *Let $f \in L^n$, then, for any $p < n$, there exists a constant $C = C(n, \Lambda, p, \Omega, a(x))$ such that*

$$\|D^2 u\|_{p,\Omega} \leq C\|f\|_{n,\Omega} \tag{1.5}$$

(c) *Let $|f|^n \ln(2 + |f|^n) \in L^1$, then there exists a constant $C = C(n, \Lambda, \Omega, a(x))$ such that*

$$\|D^2 u\|_{n,\Omega} \leq C\| \, |f|^n \ln(2 + |f|^n)\|_{1,\Omega} \tag{1.6}$$

As an application of the above estimates, we have by using the continuity method.

Theorem 1.3 *Let $a(x) \in W^{1,n}$ and $f \in L^p$ for some $p > n$. Then the homogeneous Dirichlet problem of Equation (1.2) has a unique solution u with (1.4) holding.*

Remark Let $L = \sum a_{ij}(x)D_{ij} + \sum b_i(x)D_i + c(x)$ and u be a strong solution of $Lu = f(x)$. Suppose that $b_i \in L^p$ and $c \in L^p$. Then part (a) of Theorem 1.2 and Theorem 1.3 continue to hold.

To end this section we introduce some notations.

Let Q_l denote the cube in R^n of side l and center origin and B_l, the ball of radius l and center origin. Set $Q_l(x) = x + Q_l$ and $B_l(x) = x + B_l$.

For a measurable set A, $|A|$ is the measure of A.

$C = C(*, \cdots, *)$ denotes a constant depending only on quantities appearing in parentheses. We say a constant is universal if it depends on the controlled quantities. The same letter will be used to denote different constants depending on the same set of arguments.

We will need the notation

$$\fint_\Omega = \frac{1}{|\Omega|} \int_\Omega$$